Random restricted partitions

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Abstract

We study two types of probability measures on the set of integer partitions of n with at most m parts. The first one chooses the random partition with a chance related to its largest part only. We then obtain the limiting distributions of all of the parts together and that of the largest part as n tends to infinity while m is fixed or tends to infinity. In particular, if m goes to infinity not fast enough, the largest part satisfies the central limit theorem. The second measure is very general. It includes the Dirichlet distribution and the uniform distribution as special cases. We derive the asymptotic distributions of the parts jointly and that of the largest part by taking limit of n and m in the same manner as that in the first probability measure. For one case, the largest part has the Poisson-Dirichlet distribution asymptotically.

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1 Introduction

Recall the partition κ of a positive integer n is a sequence of positive integers $k_1 \geq k_2 \geq \cdots \geq k_m$ with $m \geq 1$ whose sum is n. The number m is called the length of κ and k_i the ith largest part of κ. Let $\mathcal{P}_n$ denote the set of partitions of n and $\mathcal{P}_n(m)$ the set of partitions of n with length *at most* m. Thus $1 \leq m \leq n$ and $\mathcal{P}_n(n)$=$\mathcal{P}_n$.

The set of all partitions $\mathcal{P} = \cup_n \mathcal{P}_n$ is called the macrocanonical ensemble. The partitions of n, $\mathcal{P}_n = \cup_m \mathcal{P}_n(m)$, is called the canonical ensemble and $\mathcal{P}_n(m)$ is the microcanonical ensemble. Integer partitions have a close relationship with statistical physics (Auluck and Kothari (1946); Bohr and Kalckar (1937); Van Lier and Uhlenbeck (1937)). To be more precise, a partition $\kappa \in \mathcal{P}_n$ can be interpreted as an assembly of particles with total energy n. The number of particles is the length of κ; the number of particles with energy l is equal to $\#\{j : k_j = l\}$. Thus $\mathcal{P}_n(m)$ is the set of configurations κ with a given number of particles m. It is known that $\mathcal{P}_n(m)$ corresponds to the Bose-Einstein assembly (see section 3 in Auluck and Kothari (1946) for a brief discussion). Therefore the asymptotic distribution of a probability measure on $\mathcal{P}_n(m)$ as n tends to infinity is connected to how the total energy of the system is distributed among a given number of particles.

The most natural probability measure on $\mathcal{P}_n(m)$ is the uniform measure. The uniform measure on $\mathcal{P}_n(m)$ for $m = n$ has been well-studied (see Erdös and Lehner (1941); Fristedt (1993); Pittel (1997)). However, for the other values of m, to our best knowledge, the whole picture is not clear yet. In the authors' previous paper Jiang and Wang (2016), as a by-product of studying the eigenvalues of Laplacian-Beltrami operator defined on symmetric polynomials, the limiting distribution of $(k_1, \ldots, k_m)$ chosen uniformly from $\mathcal{P}_n(m)$ is derived for fixed integer m. This is one of the motivations resulting in this paper. As a special case of a more general measure on $\mathcal{P}_n(m)$, we obtain the asymptotic joint distribution of $(k_1, \ldots, k_m) \in \mathcal{P}_n(m)$ imposed with a uniform measure for $m \to \infty$ and $m = o(n^{1/3})$. It would be an intriguing question to understand the uniform measure on $\mathcal{P}_n(m)$ for all value of m. The limiting shape of the young diagram corresponding to $\mathcal{P}_n(m)$ with respect to uniform measure was studied in Vershik (1996); Vershik and Kerov (1985); Vershik and Yakubovich (2003) and Petrov (2009) for $m = n$ and for $m = c\sqrt{n}$ where c is a positive constant.

Another important class of probability measure on $\mathcal{P}_n(m)$ is the Plancherel measure or the more general α-Jack measure. Plancherel measure is a special case of α-Jack measure with $\alpha = 1$. It is known the both the Plancherel measure (see Baik et al. (1999); Borodin et al. (2000); Johansson (2001); Okounkov (2005), a survey by Okounkov (2000) and the references therein) and α-Jack measure (see for instance Borodin and Olshanski (2005); Fulman (2004); Matsumoto (2008)) have a deep connection with random matrix theory.

In this paper, we consider two new probability measures on $\mathcal{P}_n(m)$ assuming either m is fixed or m tends to infinity with n. We investigate the asymptotic joint distributions of

$(k_1, \ldots, k_m)$ as n tends to infinity. We first introduce the probability measures on $\mathcal{P}_n(m)$ and present the main results in Section 1.1 and 1.2. The proofs are given in the remaining of the paper.

1.1 Restricted geometric distribution

The first kind of random partitions on $\mathcal{P}_n(m)$ is defined as following: for $\kappa = (k_1, \ldots, k_m) \in \mathcal{P}_n(m)$, consider the probability measure

$$P(\kappa) = c \cdot q^{k_1} \tag{1.1}$$

where $0 < q < 1$ and $c = c_{n,m}$ is the normalizing constant that $\sum_{\kappa \in \mathcal{P}_n(m)} P(\kappa) = 1$. We call this probability measure the *restricted geometric distribution*. This probability measure favors the partitions κ with the smallest possible largest part k_1. Thus we concern the fluctuation of k_1 around $\lceil \frac{n}{m} \rceil$.

When m is a fixed integer, the main result is the following.

Theorem 1. *For given $m \geq 2$, let $\kappa = (k_1, \ldots, k_m) \in \mathcal{P}_n(m)$ be chosen with probability $P(\kappa)$ as in (1.1). For a subsequence $n \equiv j_0$ (mod m), define $j = j_0$ if $1 \leq j_0 \leq m-1$ and $j = m$ if $j_0 = 0$. Then as $n \to \infty$, we have $\left(k_1 - \lceil \frac{n}{m} \rceil, \ldots, k_m - \lceil \frac{n}{m} \rceil\right)$ converges to a discrete random vector with pmf*

$$f(l_1, \cdots, l_m) = \frac{q^{l_1}}{\sum_{l=0}^{\infty} q^l \cdot |\mathcal{P}_{m(l+1)-j}(m-1)|}$$

for all integers $(l_1, \cdots, l_m)$ with $l_1 \geq 0$, $l_1 \geq \cdots \geq l_m$ and $\sum_{i=1}^m l_i = j - m$.

From Theorem 1, we immediately obtain the limiting distribution of the largest part k_1, which fluctuates around its smallest possible value $\lceil \frac{n}{m} \rceil$. As a consequence, the conditional distribution of $(k_2, \ldots, k_m)$ given the largest part k_1 is asymptotically a uniform distribution.

Corollary 1. *Given $m \geq 2$, let $\kappa = (k_1, \ldots, k_m) \in \mathcal{P}_n(m)$ be chosen with probability $P(\kappa)$ as in (1.1). For a subsequence $n \equiv j_0$ (mod m), define $j = j_0$ if $1 \leq j_0 \leq m-1$ and $j = m$ if $j_0 = 0$. Then as $n \to \infty$, we have $k_1 - \lceil \frac{n}{m} \rceil$ converges to a discrete random variable with pmf*

$$f(l) = \frac{q^l \cdot |\mathcal{P}_{ml+m-j}(m-1)|}{\sum_{l=0}^{\infty} q^l \cdot |\mathcal{P}_{ml+m-j}(m-1)|}, \quad l \geq 0.$$

Furthermore, the conditional distribution of $(k_2 - \lceil \frac{n}{m} \rceil, \ldots, k_m - \lceil \frac{n}{m} \rceil)$ given $k_1 = \lceil \frac{n}{m} \rceil + l_1$ $(l_1 \geq 0)$ is asymptotically a uniform distribution on the set $\left\{(l_2, \ldots, l_m) \in \mathbb{Z}^{m-1};\ l_1 \geq l_2 \geq \ldots \geq l_m \text{ and } l_1 + \sum_{i=2}^m l_i = j - m\right\}$.

We present the proofs of Theorem 1 and Corollary 1 in Section 2.1.

When m tends to infinity with n and $m = o(n^{1/3})$, we consider the limiting distribution of the largest part k_1. The main result is that with proper normalization, the largest part k_1 converges to a normal distribution.

THEOREM 2. *Given $q \in (0,1)$, let $\kappa = (k_1, \ldots, k_m) \in \mathcal{P}_n(m)$ be chosen with probability $P(\kappa)$ as in (1.1). Set $\lambda = -\log q > 0$. If $m = m_n \to \infty$ with $m = o(n^{1/3})$, then $\frac{1}{\sqrt{m}}(k_1 - \lceil \frac{n}{m} \rceil - \gamma m)$ converges weakly to $N(0, \sigma^2)$ as $n \to \infty$, where*

$$\gamma = \frac{1}{\lambda^2}\int_0^\lambda \frac{t}{e^t - 1}\,dt \quad \text{and} \quad \sigma^2 = \frac{2}{\lambda^3}\int_0^\lambda \frac{t}{e^t-1}\,dt - \frac{1}{\lambda(e^\lambda - 1)} > 0.$$

The proof of Theorem 2 is analytic and quite involved. We use the Laplace method to estimate the normalization constant $c = c_{n,m}$ in (1.1). The same analysis is applied to obtain the asymptotic distribution of the largest part k_1. Thanks to the Szekeres formula (see (2.6)) for the number of restricted partitions, we first approximate $c_{n,m}$ with an integral

$$c_{n,m} \approx C(m) \cdot \int \exp(m\psi(t))\,dt$$

for some function $\psi(t)$ that has a global maximum at $t_0 > 0$. Thus

$$\psi(t) \approx \psi(t_0) - \frac{1}{2}|\psi''(t_0)|t^2$$

and

$$c_{n,m} \quad \approx \quad C(m)e^{m\psi(t_0)} \cdot \int \exp(-\frac{1}{2}m|\psi''(t_0)|)\,dt. \tag{1.2}$$

The most significant contribution in the integral comes from the t close to t_0. Indeed, the integral in (1.2) is reduced to a Gaussian integral as $n \to \infty$. We prove Theorem 2 in Section 2.2.

It remains to consider the conditional distribution of $(k_2, \ldots, k_m)$ given the largest part k_1. It is convenient to work with $k_i = \lceil \frac{n}{m} \rceil + l_i$ for $1 \le i \le m$. In view of Theorem 2, let $k_1 = \lceil \frac{n}{m} \rceil + l_1$ with $l_1 = \gamma m + C \cdot \sqrt{m}$. Given l_1, $(l_2, \ldots, l_m)$ has a uniform distribution on the set $\{(l_2, \ldots, l_m) \in \mathbb{Z}^{m-1};\, l_1 \ge l_2 \ge \ldots \ge l_m \text{ and } l_1 + \sum_{i=2}^m l_i = j - m\}$. We consider a linear transform $(j_2, \ldots, j_m) = (l_1 - l_2, \ldots, l_1 - l_m)$. Since uniform distribution is preserved under linear transformations, $(j_2, \ldots, j_m)$ has the uniform distribution on the set $\{(j_2, \ldots, j_m) \in \mathbb{N}^{m-1};\, j_m \ge \ldots \ge j_3 \ge j_2 \text{ and } \sum_{i=2}^m j_i = ml_1 + m - j\}$.

In general, the problem is reduced to understand the uniform distribution on the set

$$\{(\lambda_2, \ldots, \lambda_m) \in \mathbb{N}^{m-1};\, \lambda_2 \ge \ldots \ge \lambda_m \ge 0 \text{ and } \sum_{i=2}^m \lambda_i = ml_1\}.$$

To our best knowledge, it is not even clear what the limiting joint distribution of a partition chosen uniformly from $\mathcal{P}_{m^2}(\gamma m)$ is as m tends to infinity. We raise the following question for future projects.

Question 1. *Given $q \in (0,1)$, let $\kappa = (k_1, \ldots, k_m) \in \mathcal{P}_n(m)$ be chosen with probability $P(\kappa)$ as in (1.1). Assume m tends to infinity with n and $m = o(n^{1/3})$. Determine the asymptotic joint distribution of $(k_2, \ldots, k_m)$ given k_1. Furthermore, what is the limiting distribution of $(k_1, k_2, \ldots, k_m)$ as n tends to infinity?*

We have considered the limiting distribution of $\kappa \in \mathcal{P}_n(m)$ chosen as in (1.1) for m fixed as well as $m = o(n^{1/3})$. It is also interesting to investigate this probability measure for other ranges of m.

Question 2. *Given $q \in (0,1)$, let $\kappa = (k_1, \ldots, k_m) \in \mathcal{P}_n(m)$ be chosen with probability $P(\kappa)$ as in (1.1). Identify the asymptotic distribution of κ for the entire range $1 \le m \le n$.*

1.2 A generalized distribution

Next we consider a probability measure on $\mathcal{P}_n(m)$ by choosing a partition $\kappa = (k_1, \ldots, k_m) \vdash n$ with chance

$$P_n(\kappa) = c \cdot f(\frac{k_1}{n}, \ldots, \frac{k_m}{n}) \tag{1.3}$$

where $c = c_{n,m} = \big(\sum_{(k_1,\ldots,k_m)\in\mathcal{P}_n(m)} f(\frac{k_1}{n}, \ldots, \frac{k_m}{n})\big)^{-1}$ is the normalizing constant and $f(x_1, \ldots, x_m)$ is defined on $\overline{\nabla}_{m-1}$, the closure of ∇_{m-1}. Here ∇_{m-1} is the ordered $(m-1)$-dimensional simplex defined as

$$\nabla_{m-1} := \Big\{(y_1, \ldots, y_m) \in [0,1]^m; y_1 > y_2 > \ldots > y_{m-1} > y_m \text{ and } y_m = 1 - \sum_{i=1}^{m-1} y_i\Big\}.$$

We assume f is a probability density function on ∇_{m-1} and is Lipschitz on $\overline{\nabla}_{m-1}$.

When m is a fixed integer, we study the limiting joint distribution of the parts of κ chosen as in (1.3). The main result is the following.

Theorem 3. *Let $m \ge 2$ be a fixed integer. Assume $\kappa = (k_1, \ldots, k_m) \in \mathcal{P}_n(m)$ is chosen as in* (1.3)*, where f is a probability density function on ∇_{m-1} and f is Lipschitz on $\overline{\nabla}_{m-1}$. Then $(\frac{k_1}{n}, \ldots, \frac{k_m}{n})$ converges weakly to a probability measure μ with density function $f(y_1, \ldots, y_m)$ defined on ∇_{m-1}.*

From Theorem 3, we can immediately obtain the limiting convergence to several familiar distributions. We say $(X_1, \ldots, X_m)$ has the *symmetric Dirichlet distribution* with parameter $\alpha > 0$, denoted by $(X_1, \ldots, X_m) \sim \mathrm{Dir}(\alpha)$, if the distribution has pdf

$$\frac{\Gamma(m\alpha)}{\Gamma(\alpha)^m} x_1^{\alpha-1} \cdots x_m^{\alpha-1}$$

on the $(m-1)$-dimensional simplex

$$W_{m-1} := \Big\{(x_1, \ldots, x_{m-1}, x_m) \in [0,1]^m; \sum_{i=1}^m x_i = 1\Big\}$$

and zero elsewhere. Specially, if $\alpha = 1$, this is the uniform distribution on $\mathcal{P}_n(m)$.

Corollary 2. *Let $m \geq 2$ be a fixed integer. Assume $\kappa = (k_1, \ldots, k_m) \in \mathcal{P}_n(m)$ is chosen as in (1.3) with $f(x_1, \ldots, x_m) = c \cdot x_1^{\alpha-1} \cdots x_m^{\alpha-1}$ for some $\alpha > 2$ or $\alpha = 1$ and $1/c = \int_{\nabla_{m-1}} x_1^{\alpha-1} \cdots x_m^{\alpha-1} \, dx_1 \ldots dx_{m-1}$, then*

$$(\frac{k_1}{n}, \ldots, \frac{k_m}{n}) \to (X_{(1)}, \ldots, X_{(m)})$$

where $(X_{(1)}, \ldots, X_{(m)})$ is the decreasing order statistics of $(X_1, \ldots, X_m) \sim Dir(\alpha)$.

Corollary 3. *Let $m \geq 2$ be a fixed integer. Assume $\kappa = (k_1, \ldots, k_m) \in \mathcal{P}_n(m)$ is chosen as in (1.3) with $f(x_1, \ldots, x_m) = c \cdot x_1^{\alpha-1} \cdots x_m^{\alpha-1}$ for some $\alpha > 2$ or $\alpha = 1$ and $1/c = \int_{\nabla_{m-1}} x_1^{\alpha-1} \cdots x_m^{\alpha-1} \, dx_1 \ldots dx_{m-1}$, then*

$$\left((\frac{k_1}{n})^\alpha, \ldots, (\frac{k_m}{n})^\alpha \right) \to (x_1, \ldots, x_m)$$

as $n \to \infty$, where $(x_1, \ldots, x_m)$ has the uniform distribution on

$$\{(y_1, \ldots, y_m) \in [0,1]^m; \sum_{i=1}^m y_i^{1/\alpha} = 1, y_1 \geq \ldots \geq y_m\},$$

or equivalently, $(x_1, \ldots, x_m)$ is the decreasing order statistics of the uniform distribution on $\{(y_1, \ldots, y_m) \in [0,1]^m; \sum_{i=1}^m y_i^{1/\alpha} = 1\}$.

For the special case $\alpha = 1$, that is, κ is chosen uniformly from $\mathcal{P}_n(m)$, the conclusion of Corollary 3 is first proved in Jiang and Wang (2016). The proofs of Theorem 3, Corollary 2 and Corollary 3 are included in Section 3.1.

When m grows with n, we establish the limiting distribution of random restricted partitions in $\mathcal{P}_n(m)$. Define

$$\nabla = \{(y_1, y_2, \cdots) \in [0,1]^\infty; \ y_1 \geq y_2 \geq \cdots \text{ and} \sum_{i=1}^\infty y_i \leq 1\}.$$

Note that ∇_{m-1} can be viewed as subsets of

$$\nabla_\infty = \{(y_1, y_2, \ldots) \in [0,1]^\infty; \ y_1 \geq y_2 \geq \cdots \text{ and} \sum_{i=1}^\infty y_i = 1\}$$

by natural embedding. And ∇ is the closure of ∇_∞ in $[0,1]^\infty$ with topology inherited from $[0,1]^\infty$. By Tychonoff's theorem, ∇_{m-1} and ∇ are compact. Furthermore, both ∇_{m-1} and ∇ are compact Polish space and thus any probability measure on ∇_{m-1} is tight. Therefore, for probability measures $\{\mu_n\}_{n\geq1}$ and μ on ∇, μ_n converges to μ weakly if all the finite-dimensional distribution of μ_n converges to the corresponding finite-dimensional distribution of μ.

THEOREM 4. *Let $m = o(n^{1/3}) \to \infty$ as $n \to \infty$. Assume $\kappa = (k_1, \ldots, k_m) \in \mathcal{P}_n(m)$ is chosen with probability as in (1.3) where f is a probability density function on ∇_{m-1} and is Lipschitz on $\overline{\nabla}_{m-1}$. Let $(X_{m,1}, \cdots, X_{m,m})$ have density function $f(y_1, \cdots, y_m)$ defined on ∇_{m-1}. If $(X_{m,1}, \cdots, X_{m,m})$ converges weakly to X defined on ∇ as $n \to \infty$, then $(\frac{k_1}{n}, \cdots, \frac{k_m}{n})$ converges weakly to X as $n \to \infty$.*

We prove Theorem 4 in Section 3.2. We have investigated the limiting distribution of $\kappa \in \mathcal{P}_n(m)$ chosen as in (1.3) for both m fixed and $m = o(n^{1/3})$. It would be interesting to understand the limiting distribution of κ for other ranges of m. We leave this as an open question for future research.

QUESTION 3. *Let $\kappa = (k_1, \ldots, k_m) \in \mathcal{P}_n(m)$ be chosen with probability $P_n(\kappa)$ as in (1.3). Identify the asymptotic distribution of κ for the entire range $1 \le m \le n$.*

Notation: For $x \in \mathbb{R}$, the notation $\lceil x \rceil$ stands for the smallest integer greater than or equal to x. The symbol $[x]$ denotes the largest integer less than or equal to x. We use $\mathbb{Z}$ to be the set of all real integers. For a set A, the notation $\#A$ or $|A|$ stands for the cardinality of A. We use $c \cdot A = \{c \cdot a : a \in A\}$. Denote $\mathcal{P}_0(k) = 1$ for convenience. For $f(n), g(n) > 0$, $f(n) \sim g(n)$ if $\lim_{n\to\infty} f(n)/g(n) = 1$.

2 Proofs of restricted geometric distribution

2.1 Case I: m is fixed

We start with a lemma concerning the number of restricted partitions $\mathcal{P}_n(m)$ with the largest part fixed.

LEMMA 2.1. *Let $l \ge 0$, $m \ge 2$ and $n \ge 1$ be integers. Set $j = m + n - m\lceil \frac{n}{m} \rceil$. Then $1 \le j \le m$. If $0 \le l \le \frac{1}{m-1}(\frac{n}{m} - m)$, we have*

$$\#\Big\{(k_1, k_2, \ldots, k_m) \in \mathcal{P}_n(m);\ k_1 = \lceil \frac{n}{m} \rceil + l\Big\} = |\mathcal{P}_{m(l+1)-j}(m-1)|; \tag{2.1}$$

If $\frac{1}{m-1}(\frac{n}{m} - m) < l \le n - \lceil \frac{n}{m} \rceil$, we have

$$\#\Big\{(k_1, k_2, \ldots, k_m) \in \mathcal{P}_n(m);\ k_1 = \lceil \frac{n}{m} \rceil + l\Big\} \le |\mathcal{P}_{m(l+1)-j}(m-1)|. \tag{2.2}$$

Proof. For $\kappa = (k_1, \ldots, k_m) \in \mathcal{P}_n(m)$, let us rewrite $k_i = \lceil \frac{n}{m} \rceil + l_i$ for $1 \le i \le m$. By assumption, $l_1 = l \ge 0$. Since $\kappa \vdash n$, we have $l_1 \ge l_2 \ge \ldots \ge l_m \ge -\lceil \frac{n}{m} \rceil$ and $l_1 + \sum_{i=2}^m l_i =$

$n-m\lceil\frac{n}{m}\rceil = j-m$ by assumption. Therefore,

$$\begin{aligned}
& \#\Big\{(k_1,k_2,\dots,k_m)\in\mathcal{P}_n(m);\ k_1=\lceil\frac{n}{m}\rceil+l_1\Big\} \\
= & \#\Big\{(l_2,\dots,l_m)\in\mathbb{Z}^{m-1};\ l_1\ge l_2\ge\dots\ge l_m\ge-\lceil\frac{n}{m}\rceil \text{ and } l_1+\sum_{i=2}^m l_i=j-m\Big\} \\
= & \#\Big\{(j_2,\dots,j_m)\in\mathbb{Z}^{m-1};\ l_1+\lceil\frac{n}{m}\rceil\ge j_m\ge\dots\ge j_2\ge 0 \text{ and } \sum_{i=2}^m j_i=m(l_1+1)-j\Big\}
\end{aligned}$$

by the transform $j_i=l_1-l_i$ for $2\le i\le m$.

Assume $0\le l\le\frac{1}{m-1}(\frac{n}{m}-m)$. If $j_m\ge\dots\ge j_2\ge 0$ and $\sum_{i=2}^m j_i=m(l_1+1)-j$, then

$$j_m\le\sum_{i=2}^m j_i=m(l_1+1)-j\le m(l_1+1)\le l_1+\lceil\frac{n}{m}\rceil$$

by assumption, the notation $l_1=l$ and the fact $\lceil x\rceil\ge x$ for any $x\in\mathbb{R}$. It follows that the left hand side of (2.1) is identical to

$$\begin{aligned}
& \#\Big\{(j_2,\dots,j_m)\in\mathbb{Z}^{m-1};\ j_m\ge\dots\ge j_2\ge 0 \text{ and } \sum_{i=2}^m j_i=m(l_1+1)-j\Big\} \\
= & |\mathcal{P}_{m(l+1)-j}(m-1)|.
\end{aligned}$$

For $\frac{1}{m-1}(\frac{n}{m}-m)+1\le l\le n-\lceil\frac{n}{m}\rceil$, the upper bound (2.2) follows directly from the definitions of the sets. □

Now we are ready to present the proof of Theorem 1.

Proof of Theorem 1. First, it is easy to check that for the subsequence $n\equiv j_0 \pmod m$, if we define $j=j_0$ if $1\le j_0\le m-1$ and $j=m$ if $j_0=0$, then $j=m+n-m\lceil\frac{n}{m}\rceil$. Set

$$M_n=\Big[\frac{1}{m-1}(\frac{n}{m}-m)\Big]. \tag{2.3}$$

We first estimate the normalizing constant c in (1.1).

$$\begin{aligned}
1 &= \sum_{\kappa\in\mathcal{P}_n(m)} P(\kappa)=c\cdot\sum_{k_1=\lceil\frac{n}{m}\rceil}^{n}\ \sum_{(k_1,k_2,\dots,k_m)\vdash n} q^{k_1} \\
&= c\cdot\sum_{l=0}^{n-\lceil\frac{n}{m}\rceil} q^{\lceil\frac{n}{m}\rceil+l}\sum_{(\lceil\frac{n}{m}\rceil+l,k_2,\dots,k_m)\vdash n} 1.
\end{aligned}$$

Indeed, as n tends to infinity,

$$\begin{aligned}
& \sum_{l=0}^{n-\lceil\frac{n}{m}\rceil} q^{\lceil\frac{n}{m}\rceil+l}\sum_{(\lceil\frac{n}{m}\rceil+l,k_2,\dots,k_m)\vdash n} 1 \\
\sim & \sum_{l=0}^{M_n} q^{\lceil\frac{n}{m}\rceil+l}\sum_{(\lceil\frac{n}{m}\rceil+l,k_2,\dots,k_m)\vdash n} 1.
\end{aligned}$$

By Lemma 2.1,

$$\begin{aligned}
\frac{\sum_{M_n+1}^{n-\lceil\frac{n}{m}\rceil} q^{\lceil\frac{n}{m}\rceil+l}\sum_{(\lceil\frac{n}{m}\rceil+l,k_2,\ldots,k_m)\vdash n}1}{\sum_{l=0}^{M_n} q^{\lceil\frac{n}{m}\rceil+l}\sum_{(\lceil\frac{n}{m}\rceil+l,k_2,\ldots,k_m)\vdash n}1} &\le \frac{\sum_{M_n+1}^{n-\lceil\frac{n}{m}\rceil} q^l\cdot|\mathcal{P}_{m(l+1)-j}(m-1)|}{\sum_{l=0}^{M_n} q^l\cdot|\mathcal{P}_{m(l+1)-j}(m-1)|}\\
&\sim \frac{\sum_{M_n+1}^{n-\lceil\frac{n}{m}\rceil} q^l\binom{ml+m-j-1}{m-2}}{\sum_{l=0}^{M_n} q^l\binom{ml+m-j-1}{m-2}},
\end{aligned}$$

where the last equality follows from (3.1). Note that the series $\sum_{s=1}^{\infty} s^{m-2}q^s$ converges for $0<q<1$. We have

$$\frac{\sum_{M_n+1}^{n-\lceil\frac{n}{m}\rceil} q^l\binom{ml+m-j-1}{m-2}}{\sum_{l=0}^{M_n} q^l\binom{ml+m-j-1}{m-2}} = O(\frac{\sum_{M_n+1}^{n-\lceil\frac{n}{m}\rceil} q^l l^{m-2}}{\sum_{l=0}^{M_n} q^l l^{m-2}}) = o(1).$$

Therefore, one obtains the normalizing constant

$$c \sim \frac{1}{q^{\lceil\frac{n}{m}\rceil}\sum_{l=0}^{M_n} q^l\cdot|\mathcal{P}_{m(l+1)-j}(m-1)|}. \tag{2.4}$$

Now we study the limiting joint distribution of the parts

$$(k_1,k_2,\ldots,k_m) = (\lceil\frac{n}{m}\rceil+l_1, \lceil\frac{n}{m}\rceil+l_2,\ldots,\lceil\frac{n}{m}\rceil+l_m).$$

First, we claim that it is enough to consider l_1 to be any fixed integer from $\{0,1,2,\ldots\}$. Indeed, for any $L=L(n)\to\infty$ as $n\to\infty$, it follows from (3.1) that

$$\begin{aligned}
P(k_1\ge\lceil\frac{n}{m}\rceil+L) &= \sum_{l=L}^{M_n} P(k_1=\lceil\frac{n}{m}\rceil+l)\\
&= \sum_{l=L}^{M_n} c\cdot q^{\lceil\frac{n}{m}\rceil+l}|\mathcal{P}_{ml+m-j}(m-1)|\\
&\sim c\cdot q^{\lceil\frac{n}{m}\rceil}\sum_{l=L}^{M_n}\frac{\binom{ml+m-j-1}{m-2}}{(m-1)!}q^l.
\end{aligned}$$

Plugging in the normalizing constant $c_{n,m}$ and let $L\to\infty$, we have

$$\begin{aligned}
P(k_1\ge\lceil\frac{n}{m}\rceil+L) &= O\Big(\frac{\sum_{l=L}^{M_n} l^{m-2}q^l}{\sum_{l=0}^{M_n} q^l\cdot|\mathcal{P}_{ml+m-j}(m-1)|}\Big)\\
&= o(1),
\end{aligned}$$

as $n\to\infty$. The last equality follows from the fact that the series $\sum_{s=1}^{\infty} s^{m-2}q^s$ converges for $0<q<1$.

Likewise, we have as n tends to infinity,

$$c \sim q^{-\lceil\frac{n}{m}\rceil}\frac{1}{\sum_{l=0}^{\infty} q^l\cdot|\mathcal{P}_{ml+m-j}(m-1)|}. \tag{2.5}$$

Therefore, for any given $l_1 = 0, 1, 2, \ldots$, we conclude that

$$\begin{aligned} & P\Big(k_1 = \lceil \frac{n}{m} \rceil + l_1, k_2 = \lceil \frac{n}{m} \rceil + l_2, \ldots, k_m = \lceil \frac{n}{m} \rceil + l_m\Big) \\ = \quad & c \cdot q^{\lceil \frac{n}{m} \rceil + l_1} \to \frac{q^{l_1}}{\sum_{l=0}^{\infty} q^l \cdot |\mathcal{P}_{ml+m-j}(m-1)|}. \end{aligned}$$

□

Proof of Corollary 1. By Theorem 1, it is enough to consider $k_1 = \lceil \frac{n}{m} \rceil + l$ for $l \in \{0, 1, 2, \ldots\}$ in the limiting distribution. From (1.1), Lemma 2.1 and (2.5),

$$\begin{aligned} P(k_1 = \lceil \frac{n}{m} \rceil + l) &= c \cdot q^{\lceil \frac{n}{m} \rceil + l} \sum_{(\lceil \frac{n}{m} \rceil + l, k_2, \ldots, k_m) \vdash n} 1 \\ &= c \cdot q^{\lceil \frac{n}{m} \rceil + l} \cdot |\mathcal{P}_{ml_1+m-j}(m-1)| \\ &\to \frac{q^l \cdot |\mathcal{P}_{ml+m-j}(m-1)|}{\sum_{l=0}^{\infty} q^l \cdot |\mathcal{P}_{ml+m-j}(m-1)|} \end{aligned}$$

as $n \to \infty$.

Furthermore, since

$$\begin{aligned} & P(k_2 - \lceil \frac{n}{m} \rceil = l_2, \ldots, k_m - \lceil \frac{n}{m} \rceil = l_m \mid k_1 - \lceil \frac{n}{m} \rceil = l_1) \\ & = \frac{P(k_1 - \lceil \frac{n}{m} \rceil = l_1, k_2 - \lceil \frac{n}{m} \rceil = l_2, \ldots, k_m - \lceil \frac{n}{m} \rceil = l_m)}{P(k_1 - \lceil \frac{n}{m} \rceil = l_1)} \\ & \sim \frac{f(l_1, \ldots, l_m)}{f(l_1)} = \frac{1}{|\mathcal{P}_{ml_1+m-j}(m-1)|} \end{aligned}$$

as $n \to \infty$, it follows immediately the conditional distribution of $(k_2 - \lceil \frac{n}{m} \rceil, \ldots, k_m - \lceil \frac{n}{m} \rceil)$ given $k_1 = \lceil \frac{n}{m} \rceil + l_1$ $(l_1 \geq 0)$ is asymptotically a uniform distribution on the set $\big\{(l_2, \ldots, l_m) \in \mathbb{Z}^{m-1}; l_1 \geq l_2 \geq \ldots \geq l_m \text{ and } l_1 + \sum_{i=2}^m l_i = j - m\big\}$. This completes the proof. □

2.2 Case II: m tends to infinity and $m = o(n^{1/3})$

Szekeres formula (see Szekeres (1951, 1953); see also Canfield (1997) and Romik (2005)) says that

$$|\mathcal{P}_n(k)| \sim \frac{f(u)}{n} e^{\sqrt{n} g(u)} \tag{2.6}$$

uniformly for $k \geq n^{1/6}$, where $u = k/\sqrt{n}$, and

$$f(u) = \frac{v}{2^{3/2} \pi u} \big(1 - e^{-v} - \frac{1}{2} u^2 e^{-v}\big)^{-1/2}; \tag{2.7}$$

$$g(u) = \frac{2v}{u} - u \log(1 - e^{-v}), \tag{2.8}$$

with $v = v(u)$ determined implicitly by

$$u^2 = \frac{v^2}{\int_0^v \frac{t}{e^t - 1}\, dt}. \tag{2.9}$$

We start with a technical lemma that will be used in the proof of Theorem 2 later.

Lemma 2.2. *Let $\lambda > 0$ be given. Define $\psi(t) = \frac{g(t)}{t} - \frac{\lambda}{t^2}$ for $t > 0$. Then*

$$t_0 := \frac{\lambda}{(\int_0^\lambda \frac{t}{e^t-1}\, dt)^{1/2}} \quad \textit{satisfies} \quad \psi''(t_0) = -\frac{2\lambda(e^\lambda - 1)}{t_0^4(e^\lambda - 1 - \frac{1}{2}t_0^2)} < 0.$$

Further, $\psi'(t_0) = 0$, $\psi(t)$ is strictly increasing on $(0, t_0]$ and strictly decreasing on $[t_0, \infty)$.

Proof. Trivially, the function $\frac{t}{e^t-1} = (\sum_{i=1}^\infty \frac{t^{i-1}}{i!})^{-1}$ is positive and decreasing in $t \in (0, \infty)$. It follows that $v = v(u) > 0$ for all $u \in (0, \infty)$ and

$$\frac{v^2}{u^2} = \int_0^v \frac{t}{e^t - 1}\, dt > \frac{v^2}{e^v - 1}.$$

Thus $e^v - 1 - u^2 > 0$. In particular,

$$e^v - 1 - \frac{1}{2}u^2 > 0. \tag{2.10}$$

By taking derivative from (2.9), we get

$$2v \cdot v' = 2u \int_0^v \frac{t}{e^t - 1}\, dt + u^2 \frac{v \cdot v'}{e^v - 1}.$$

This implies that $\frac{v'}{e^v-1} = \frac{2v'}{u^2} - \frac{2v}{u^3}$, or equivalently,

$$v' = \frac{v}{u} + \frac{uv}{2(e^v - 1 - \frac{1}{2}u^2)}. \tag{2.11}$$

Consequently, $v' = v'(u) > 0$ for all $u > 0$, and thus $v(u)$ is strictly increasing on $(0, \infty)$. Take derivative on $g(u)$ in (2.8), and use (2.9) and (2.11) to see

$$\begin{aligned} g'(u) &= -\log(1 - e^{-v}); \\ g''(u) &= -\frac{v'e^{-v}}{1 - e^{-v}} = -\frac{v/u}{e^v - 1 - \frac{1}{2}u^2}. \end{aligned} \tag{2.12}$$

Therefore

$$(\frac{g(u)}{u})' = \frac{ug'(u) - g(u)}{u^2} \tag{2.13}$$

and

$$\begin{aligned} (\frac{g(u)}{u})'' &= \frac{g''(u)}{u} - 2\frac{g'(u)}{u^2} + 2\frac{g(u)}{u^3} \\ &= \frac{v}{u^4}\Big(4 - \frac{u^2}{e^v - 1 - \frac{1}{2}u^2}\Big). \end{aligned}$$

With the above preparation, we now study $\psi(t)$ (we switch the variable "u" to "t").

$$\begin{aligned}\psi''(t) &= (\frac{g(t)}{t} - \frac{\lambda}{t^2})'' \\ &= \frac{v}{t^4}\Big(4 - \frac{t^2}{e^v - 1 - \frac{1}{2}t^2}\Big) - \frac{6\lambda}{t^4} \\ &= \frac{1}{t^4}\Big(4v - 6\lambda - \frac{v \cdot t^2}{e^v - 1 - \frac{1}{2}t^2}\Big).\end{aligned} \tag{2.14}$$

The assertions in (2.12) and (2.13) imply

$$\Big(\frac{g(t)}{t}\Big)' = \frac{-t^2\log(1-e^{-v}) - tg(t)}{t^3} = -\frac{2v}{t^3}.$$

Thus, $\psi'(t) = \frac{2(\lambda - v)}{t^3}$. Thus, the stable point t_0 of $\psi(t)$ satisfies that $v(t_0) = \lambda$. This implies that $\psi(t)$ is strictly increasing on $(0, t_0]$ and strictly decreasing on $[t_0, \infty)$. It is not difficult to see from (2.9) that

$$t_0 = \frac{\lambda}{(\int_0^\lambda \frac{t}{e^t-1}\,dt)^{1/2}}.$$

Plug this into (2.14) to get

$$\begin{aligned}\psi''(t_0) &= -\frac{1}{t_0^4}\Big(2\lambda + \frac{\lambda \cdot t_0^2}{e^\lambda - 1 - \frac{1}{2}t_0^2}\Big) \\ &= -\frac{2\lambda(e^\lambda - 1)}{t_0^4(e^\lambda - 1 - \frac{1}{2}t_0^2)} < 0\end{aligned}$$

by (2.10). $\square$

Proof of Theorem 2. Let $M_n = \left[\frac{1}{m-1}(\frac{n}{m} - m)\right]$ as in (2.3). The assumption $m = o(n^{1/3})$ implies

$$\lim_{n\to\infty} \frac{M_n}{m} = \infty. \tag{2.15}$$

Similar to (2.4), we first claim that the normalization constant

$$c \sim \frac{1}{q^{\lceil \frac{n}{m} \rceil} \sum_{l=0}^{M_n} q^l \cdot |\mathcal{P}_{m(l+1)-j}(m-1)|}.$$

Indeed, from Lemma 2.1,

$$\begin{aligned}\frac{1}{c} &= \sum_{l=0}^{n-\lceil \frac{n}{m} \rceil} q^{\lceil \frac{n}{m} \rceil + l} \sum_{(\lceil \frac{n}{m} \rceil + l, k_2, \dots, k_m) \vdash n} 1 \\ &= \sum_{l=0}^{M_n} q^{\lceil \frac{n}{m} \rceil + l} \cdot |\mathcal{P}_{m(l+1)-j}(m-1)| + \sum_{l=M_n+1}^{n-\lceil \frac{n}{m} \rceil} q^{\lceil \frac{n}{m} \rceil + l} \sum_{(\lceil \frac{n}{m} \rceil + l, k_2, \dots, k_m) \vdash n} 1\end{aligned}$$

and

$$\begin{aligned}
\sum_{l=M_n+1}^{n-\lceil\frac{n}{m}\rceil} q^{\lceil\frac{n}{m}\rceil+l} \sum_{(\lceil\frac{n}{m}\rceil+l,k_2,\ldots,k_m)\vdash n} 1 &\leq \sum_{l=M_n+1}^{n-\lceil\frac{n}{m}\rceil} q^{\lceil\frac{n}{m}\rceil+l}\cdot|\mathcal{P}_{m(l+1)-j}(m-1)| \\
&= \sum_{l=M_n+2}^{n-\lceil\frac{n}{m}\rceil+1} q^{\lceil\frac{n}{m}\rceil+l}\cdot|\mathcal{P}_{lm-j}(m-1)|.
\end{aligned}$$

Observe that $|\mathcal{P}_{lm-j}(m-1)|\leq|\mathcal{P}_{lm}(lm)|\leq e^{K\sqrt{lm}}$ for some constant $K>0$ by Hardy-Ramanujan formula. Therefore,

$$\begin{aligned}
\sum_{l=M_n+1}^{n-\lceil\frac{n}{m}\rceil} q^{\lceil\frac{n}{m}\rceil+l} \sum_{(\lceil\frac{n}{m}\rceil+l,k_2,\ldots,k_m)\vdash n} 1 &\leq q^{\lceil\frac{n}{m}\rceil}\sum_{l=M_n}^{\infty} e^{-\lambda l+K\sqrt{lm}} \\
&\leq q^{\lceil\frac{n}{m}\rceil}\sum_{l=M_n}^{\infty} e^{-\lambda l/2}\leq q^{\lceil\frac{n}{m}\rceil}\frac{e^{-\lambda M_n/2}}{1-e^{-\lambda/2}} \\
&= o\Big(\sum_{l=0}^{M_n} q^{\lceil\frac{n}{m}\rceil+l}\cdot|\mathcal{P}_{m(l+1)-j}(m-1)|\Big)
\end{aligned}$$

for n sufficiently large.

Hence, without loss of generality, we have

$$P\big(k_1=\lceil\frac{n}{m}\rceil+l\big)=\frac{q^l|\mathcal{P}_{m(l+1)-j}(m-1)|}{\sum_{l=0}^{M_n} q^l\cdot|\mathcal{P}_{m(l+1)-j}(m-1)|}$$

for $l=0,1,2,\cdots,M_n$, where $j=m+n-m\lceil\frac{n}{m}\rceil$ and $1\leq j\leq m$. Thus,

$$P\big(k_1\leq\lceil\frac{n}{m}\rceil+m\xi\big)=\frac{\sum_{l=1}^{[m\xi]+1} q^l\cdot|\mathcal{P}_{lm-j}(m-1)|}{\sum_{l=1}^{M_n+1} q^l\cdot|\mathcal{P}_{lm-j}(m-1)|} \tag{2.16}$$

for any $\xi\geq 0$.

In the following, we first apply a fine analysis to estimate the denominator

$$\sum_{l=1}^{M_n+1} q^l\cdot|\mathcal{P}_{lm-j}(m-1)|.$$

We divide the range of summation into five parts: $1\leq l\leq cm$, $Cm\leq l\leq M_n$, $cm\leq l<\gamma m-\sqrt{m}\log m$, $\gamma m+\sqrt{m}\log m<l\leq Cm$ and $\gamma m-\sqrt{m}\log m\leq l\leq\gamma m+\sqrt{m}\log m$ for some proper constants $c,C>0$ and $\gamma=t_0^{-2}$ (recall t_0 in Lemma 2.2). The most significant contribution in the summation comes from the range $\gamma m-\sqrt{m}\log m\leq l\leq\gamma m+\sqrt{m}\log m$ and others are negligible. The estimation for the numerator is similar.

Step 1: Two rough tails are negligible. First, by Hardy-Ramanujan formula, there exists a constant $K>0$ such that

$$|\mathcal{P}_{lm-j}(m-1)|\leq|\mathcal{P}_{lm}(lm)|\leq e^{K\sqrt{lm}}$$

for $l \geq 1$ as n is large. Set $\lambda = -\log q > 0$. It follows that

$$\sum_{l=Cm}^{M_n+1} q^l \cdot |\mathcal{P}_{lm-j}(m-1)| \leq \sum_{l=Cm}^{\infty} e^{-\lambda l + K\sqrt{ml}} \leq \sum_{l=Cm}^{\infty} e^{-\lambda l/2} \leq \frac{1}{1-e^{-\lambda/2}} \tag{2.17}$$

for all $l \geq (\frac{4K^2}{\lambda^2})m$, which is satisfied if $C > \frac{4K^2}{\lambda^2}$. Similarly, for the same K as above,

$$\begin{aligned}\sum_{l=1}^{cm} q^l \cdot |\mathcal{P}_{lm-j}(m-1)| &\leq \sum_{l=1}^{cm} q^l \cdot |\mathcal{P}_{[cm^2]}(m)| \\ &\leq (cm) \cdot |\mathcal{P}_{[cm^2]}(m)| \leq (cm) \cdot e^{\sqrt{c}Km} \end{aligned} \tag{2.18}$$

for all $c > 0$ as n is sufficiently large.

In the rest of the proof, the variable n will be hidden in $m = m_n$ and $j = j_n$. Keep in mind that m is sufficiently large when we say "n is sufficiently large". We set two parameters

$$C = \max\{\frac{8K^2}{\lambda^2}, 2\gamma\}; \tag{2.19}$$

$$c = \min\{\frac{\psi(t_0)^2}{16K^2}, \frac{\gamma}{2}\}. \tag{2.20}$$

Step 2: Two refined tails are negligible. Recall t_0 in Lemma 2.2. Define $\gamma = t_0^{-2}$ and

$$\begin{aligned} &\Omega_1 = \{l \in \mathbb{N};\, cm \leq l < \gamma m - \sqrt{m}B\}, \quad \Omega_2 = \{l \in \mathbb{N};\, \gamma m - \sqrt{m}B \leq l \leq \gamma m + \sqrt{m}B\}, \\ &\Omega_3 = \{l \in \mathbb{N};\, \gamma m + \sqrt{m}B < l \leq Cm\}, \end{aligned} \tag{2.21}$$

with $B = \log m$, where $c \in (0, \gamma)$ and $C > \gamma$ by (2.20) and (2.19). The limit in (2.15) asserts that $\Omega_2 \subset \{1, 2, \cdots, M_n\}$ as n is large. Then

$$\sum_{l=cm}^{Cm} q^l \cdot |\mathcal{P}_{lm-j}(m-1)| = \sum_{i=1}^{3} \sum_{l \in \Omega_i} q^l \cdot |\mathcal{P}_{lm-j}(m-1)|. \tag{2.22}$$

Easily,

$$\sum_{l \in \Omega_1 \cup \Omega_3} q^l \cdot |\mathcal{P}_{lm-j}(m-1)| \leq \sum_{l \in \Omega_1 \cup \Omega_3} q^l \cdot |\mathcal{P}_{lm}(m)| \tag{2.23}$$

Take $n = lm$ and $k = m$ in (2.6), we get

$$|\mathcal{P}_{lm}(m)| \sim \frac{f(u)}{lm} e^{\sqrt{lm}g(u)} \tag{2.24}$$

uniformly for all $cm \leq l \leq Cm$ where $u = (\frac{m}{l})^{1/2}$. Notice

$$q^l \cdot |\mathcal{P}_{lm}(m)| \sim \frac{f(u)}{lm} e^{-\lambda l + \sqrt{lm}g(u)}.$$

Consider function $-\lambda x + \sqrt{xm} \cdot g\big((mx^{-1})^{1/2}\big)$ for $x \in [cm, Cm]$. Set $t = t_x = (mx^{-1})^{1/2}$. Then

$$\begin{aligned} -\lambda x + \sqrt{xm} \cdot g\big((mx^{-1})^{1/2}\big) &= -\frac{\lambda m}{t^2} + m\frac{g(t)}{t} \\ &= m\Big(\frac{g(t)}{t} - \frac{1}{t^2}\lambda\Big). \end{aligned} \tag{2.25}$$

By (2.7) and (2.8), $f(x)$ is a continuous function on $[C^{-1/2}, c^{-1/2}]$. Therefore, $\frac{f((mj^{-1})^{1/2})}{mj} = O(m^{-2})$ uniformly for all $j \in \Omega_1 \cup \Omega_3$, which together with (2.23) yields

$$\begin{aligned} &\sum_{l \in \Omega_1 \cup \Omega_3} q^l \cdot |\mathcal{P}_{lm-j}(m-1)| \\ \leq\ & O\Big(\frac{1}{m^2}\Big) \sum_{l \in \Omega_1 \cup \Omega_3} \exp\Big[m\Big(\frac{g(t_l)}{t_l} - \frac{\lambda}{t_l^2}\Big)\Big] \\ \leq\ & O\Big(\frac{1}{m}\Big) \cdot \exp\Big[m \max_{l \in \Omega_1 \cup \Omega_3} \Big(\frac{g(t_l)}{t_l} - \frac{\lambda}{t_l^2}\Big)\Big]. \end{aligned} \tag{2.26}$$

Now

$$\max_{l \in \Omega_1 \cup \Omega_3} \Big(\frac{g(t_l)}{t_l} - \frac{\lambda}{t_l^2}\Big) = \max_{l \in \Omega_1 \cup \Omega_3} \Big\{\psi\Big(\sqrt{\frac{m}{l}}\Big)\Big\}.$$

Evidently,

$$\begin{aligned} &\Big\{\sqrt{\frac{m}{l}},\ l \in \Omega_1\Big\} \subset \Big[\Big(\frac{m}{\gamma m - \sqrt{m}\log m}\Big)^{1/2}, \frac{1}{\sqrt{c}}\Big] \subset (t_0, \infty); \\ &\Big\{\sqrt{\frac{m}{l}},\ l \in \Omega_3\Big\} \subset \Big[\frac{1}{\sqrt{C}}, \Big(\frac{m}{\gamma m + \sqrt{m}\log m}\Big)^{1/2}\Big] \subset (0, t_0). \end{aligned}$$

Recall Lemma 2.2, $\psi(t) = \frac{g(t)}{t} - \frac{\lambda}{t^2}$ is increasing $(0, t_0]$ and decreasing in $[t_0, \infty)$. It follows that

$$\begin{aligned} &\max_{l \in \Omega_1 \cup \Omega_3} \Big(\frac{g(t_l)}{t_l} - \frac{\lambda}{t_l^2}\Big) \\ \leq\ & \max\Big\{\psi\Big(\frac{\sqrt{m}}{\sqrt{\gamma m - \sqrt{m}\log m}}\Big), \psi\Big(\frac{\sqrt{m}}{\sqrt{\gamma m + \sqrt{m}\log m}}\Big)\Big\}. \end{aligned}$$

Notice

$$\begin{aligned} \Big(\frac{\sqrt{m}}{\sqrt{\gamma m \pm \sqrt{m}\log m}} - t_0\Big)^2 &= \Big[\frac{1}{\sqrt{\gamma}}\Big(1 \pm \frac{\log m}{\gamma\sqrt{m}}\Big)^{-1/2} - t_0\Big]^2 \\ &= \frac{(\log m)^2}{4\gamma^3 m}(1 + o(1)). \end{aligned}$$

From (2.32), we see that

$$\psi\Big(\frac{\sqrt{m}}{\sqrt{\gamma m \pm \sqrt{m}\log m}}\Big) = \psi(t_0) - L\frac{(\log m)^2}{m} + O(m^{-3/2}(\log m)^3)$$

as n is large, where $L = \frac{|\psi''(t_0)|}{8\gamma^3} > 0$. This joins (2.26) to yield that

$$\frac{1}{\sqrt{m}} e^{-m\psi(t_0)} \sum_{l\in\Omega_1\cup\Omega_3} q^l \cdot |\mathcal{P}_{lm-j}(m-1)| \le e^{-(L/2)(\log m)^2}$$

and thus

$$\sum_{l\in\Omega_1\cup\Omega_3} q^l \cdot |\mathcal{P}_{lm-j}(m-1)| \le \sqrt{m} e^{m\psi(t_0)-(L/2)(\log m)^2} \tag{2.27}$$

as n is large.

Step 3. The estimate of $\sum_{j\in\Omega_2}$. Take $n = lm - j$ and $k = m-1$ in (2.6), we get

$$|\mathcal{P}_{ml-j}(m-1)| \sim \frac{f(u)}{ml-j} e^{\sqrt{ml-j}\, g(u)}$$

uniformly for all $cm \le l \le Cm$ where $u = \frac{m-1}{\sqrt{lm-j}}$. By continuity,

$$\frac{f(u)}{ml-j} \sim t_0^2 f(t_0) \cdot \frac{1}{m^2} \tag{2.28}$$

uniformly for all $l \in \Omega_2$. Consequently,

$$\begin{aligned} & \sum_{l\in\Omega_2} q^l \cdot |\mathcal{P}_{lm-j}(m-1)| \\ = \quad & (1+o(1)) \frac{t_0^2 f(t_0)}{m^2} \sum_{l\in\Omega_2} \exp\Big\{ -\lambda l + \sqrt{lm-j} \cdot g\Big(\frac{m-1}{\sqrt{lm-j}}\Big)\Big\} \\ \sim \quad & \frac{t_0^2 f(t_0)}{m^2} e^{-\lambda j/m} \sum_{l\in\Omega_2} \exp\Big\{ -\frac{\lambda (m-1)^2}{m t_l^2} + \frac{m-1}{t_l} g(t_l)\Big\} \end{aligned} \tag{2.29}$$

by setting $t_x = (m-1)/\sqrt{mx-j}$ for $x \ge 2$ (recall $1 \le j \le m$), and hence $x = \frac{j}{m} + \frac{(m-1)^2}{m t_x^2}$. It is easy to verify that

$$\max_{l\in\Omega_2} |t_l - t_0| = O\Big(\frac{\log m}{\sqrt{m}}\Big) \tag{2.30}$$

as $n \to \infty$. We then have

$$\begin{aligned} & \sum_{l\in\Omega_2} q^l \cdot |\mathcal{P}_{lm-j}(m-1)| \\ \sim \quad & \frac{t_0^2 f(t_0)}{m^2} e^{\lambda t_0^{-2} - (\lambda j/m)} \sum_{l\in\Omega_2} \exp\Big\{(m-1)\big(\frac{g(t_l)}{t_l} - \frac{\lambda}{t_l^2}\big)\Big\}. \end{aligned} \tag{2.31}$$

Recall Lemma 2.2. Since $\psi'(t_0) = 0$ and $\psi''(t_0) < 0$, it is seen from the Taylor's expansion and (2.30) that

$$\psi(t_x) = \psi(t_0) + \frac{1}{2}\psi''(t_0)(t_x - t_0)^2 + O(m^{-3/2}(\log m)^3) \tag{2.32}$$

uniformly for all $x \in \Omega_2$. It follows that

$$
\begin{aligned}
& \sum_{l\in\Omega_2} \exp\Big[(m-1)\Big(\frac{g(t_l)}{t_l} - \frac{\lambda}{t_l^2}\Big)\Big] \\
= \;& (1+o(1))\cdot e^{(m-1)\psi(t_0)} \sum_{l\in\Omega_2} \exp\Big[\frac{1}{2}\psi''(t_0)(t_l-t_0)^2 m\Big].
\end{aligned}
$$

It is trivial to check that

$$
\frac{m-1}{\sqrt{mx-j}} = \frac{m-1}{\sqrt{mx}} + \frac{j}{2\gamma^{3/2}m^2} + O\big(\frac{\log m}{m^2}\big)
$$

uniformly for all $x \in \Omega_2$. Therefore,

$$
\begin{aligned}
m\Big(\frac{m-1}{\sqrt{mx-j}} - t_0\Big)^2 &= m\Big(\frac{m-1}{\sqrt{mx}} - t_0\Big)^2 + \frac{j}{\gamma^{3/2}m}\Big(\frac{m-1}{\sqrt{mx}} - t_0\Big) + O\big(\frac{\log m}{\sqrt{m}}\big) \\
&= m\Big(\frac{m-1}{\sqrt{mx}} - t_0\Big)^2 + o(1)
\end{aligned}
$$

uniformly for all $x \in \Omega_2$ by (2.30). This tells us that

$$
\begin{aligned}
& \sum_{l\in\Omega_2} \exp\Big[(m-1)\Big(\frac{g(t_l)}{t_l} - \frac{\lambda}{t_l^2}\Big)\Big] \\
= \;& (1+o(1))\cdot e^{(m-1)\psi(t_0)} \sum_{l\in\Omega_2} \exp\Big[\frac{1}{2}\psi''(t_0)\Big(\frac{m-1}{\sqrt{ml}} - t_0\Big)^2 m\Big]. \qquad (2.33)
\end{aligned}
$$

Set $a_m = \gamma m - \sqrt{m}\log m$, $b_m = \gamma m + \sqrt{m}\log m$, $c_m = (m-1)/\sqrt{m}$ and

$$
\rho(x) = \exp\Big[\frac{1}{2}\psi''(t_0)\big(\frac{c_m}{\sqrt{x}} - t_0\big)^2 m\Big] \qquad (2.34)
$$

for $x > 0$. It is easy to check that there exists an absolute constant $C_1 > 0$ such that

$$
\rho(x) \le e^{-C_1(\log m)^2} \qquad (2.35)
$$

for all $x \in (a_m, b_m)\backslash([a_m]+2, [b_m]-2)$. Hence

$$
\int_{a_m}^{b_m} \rho(x)\,dx \;=\; \Big(\sum_{l=[a_m]}^{[b_m]-1} \int_l^{l+1} \rho(x)\,dx\Big) + \epsilon_m \qquad (2.36)
$$

where $|\epsilon_m| \le e^{-C_1(\log m)^2}$ for large m. By the expression $\rho(x) = \exp\big[\frac{1}{2}\psi''(t_0)\big(\frac{c_m}{\sqrt{x}} - t_0\big)^2 m\big]$, we get

$$
\rho'(x) = -\frac{1}{2}\rho(x)\psi''(t_0)\Big(\frac{c_m}{\sqrt{x}} - t_0\Big)\frac{mc_m}{x^{3/2}}
$$

for $x > 0$. Easily, $\frac{mc_m}{x^{3/2}} = O(1)$ and $\frac{c_m}{\sqrt{x}} - t_0 = O(\frac{\log m}{\sqrt{m}})$ uniformly for all $[a_m] \le x \le [b_m]$. Thus,

$$|\rho'(x)| \le \frac{(\log m)^2}{\sqrt{m}}\rho(x)$$

for all $[a_m] \le x \le [b_m]$. Therefore, by integration by parts,

$$\begin{aligned}\Big|\int_l^{l+1}\rho(x)\,dx - \rho(l)\Big| = \Big|\int_l^{l+1}\rho'(x)(l+1-x)\,dx\Big| &\le \int_l^{l+1}|\rho'(x)|\,dx \\ &\le \frac{(\log m)^2}{\sqrt{m}}\int_l^{l+1}\rho(x)\,dx\end{aligned}$$

as m is sufficiently large. This, (2.35) and (2.36) imply

$$\Big|\sum_{l\in\Omega_2}\rho(l) - \int_{a_m}^{b_m}\rho(x)\,dx\Big| \le \frac{(\log m)^2}{\sqrt{m}}\Big(\int_{a_m}^{b_m}\rho(x)\,dx\Big) + e^{-C_1(\log m)^2}. \tag{2.37}$$

Set $\gamma_m = (\log m)\gamma^{-3/2}/2$. We see from (2.33) and (2.34) that

$$\begin{aligned}\int_{a_m}^{b_m}\rho(x)\,dx &= \frac{2c_m^2}{\sqrt{m}}\int_{-\gamma_m+o(1)}^{\gamma_m+o(1)}\Big(-\frac{u}{\sqrt{m}}+t_0\Big)^{-3}e^{\frac{1}{2}\psi''(t_0)u^2}\,du \\ &= (1+o(1))\frac{2\sqrt{m}}{t_0^3}\int_{-\gamma_m}^{\gamma_m}e^{\frac{1}{2}\psi''(t_0)u^2}\,du \\ &= (1+o(1))\frac{2\sqrt{m}}{t_0^3}\int_{-\infty}^{\infty}e^{\frac{1}{2}\psi''(t_0)u^2}\,du \\ &\sim \sqrt{m}\cdot\frac{1}{t_0^3}\sqrt{\frac{8\pi}{|\psi''(t_0)|}}\end{aligned}$$

by making the transform $u = -\big(\frac{c_m}{\sqrt{x}} - t_0\big)\sqrt{m}$. Combining this, (2.33) and (2.37), we arrive at

$$\begin{aligned}e^{-(m-1)\psi(t_0)}\sum_{l\in\Omega_2}\exp\Big[(m-1)\Big(\frac{g(t_l)}{t_l}-\frac{\lambda}{t_l^2}\Big)\Big] &= (1+o(1))\sum_{l\in\Omega_2}\rho(l) \\ &\sim \sqrt{m}\cdot\frac{1}{t_0^3}\sqrt{\frac{8\pi}{|\psi''(t_0)|}}\end{aligned} \tag{2.38}$$

as n is sufficiently large. This and (2.31) yield

$$\begin{aligned}&\sum_{l\in\Omega_2}q^l\cdot|\mathcal{P}_{lm-j}(m-1)| \\ \sim\ &\frac{t_0^2 f(t_0)}{m^2}e^{\lambda t_0^{-2}-(\lambda j/m)}\sum_{l\in\Omega_2}\exp\Big\{(m-1)\big(\frac{g(t_l)}{t_l}-\frac{\lambda}{t_l^2}\big)\Big\} \\ \sim\ &\frac{f(t_0)e^{\lambda t_0^{-2}-\psi(t_0)-(\lambda j/m)}}{t_0}\cdot\sqrt{\frac{8\pi}{|\psi''(t_0)|}}\cdot\frac{e^{m\psi(t_0)}}{m^{3/2}}\end{aligned} \tag{2.39}$$

as $m \to \infty$.

Step 4. Wrap-up of the denominator. By the choice of c in (2.20), we have $\sqrt{c} \le (4K)^{-1}\psi(t_0)$ in (2.18). Therefore we get from (2.17) that

$$\Big(\sum_{l=1}^{cm} + \sum_{l=Cm}^{M_n+1}\Big) q^l \cdot |\mathcal{P}_{lm-j}(m-1)| \le e^{\psi(t_0)m/2} \tag{2.40}$$

as n is large. This and (2.22) imply

$$\sum_{l=1}^{M_n+1} q^l \cdot |\mathcal{P}_{lm}(m-1)| = O\big(e^{\psi(t_0)m/2}\big) + \sum_{i=1}^{3}\sum_{l\in\Omega_i} q^l \cdot |\mathcal{P}_{lm-j}(m-1)|$$

as $m \to \infty$. This identity together with (2.27) and (2.39) concludes that

$$\sum_{l=1}^{M_n+1} q^l \cdot |\mathcal{P}_{lm-j}(m-1)| \sim \frac{f(t_0)e^{\lambda t_0^{-2}-\psi(t_0)-(\lambda j/m)}}{t_0} \cdot \sqrt{\frac{8\pi}{|\psi''(t_0)|}} \cdot \frac{e^{m\psi(t_0)}}{m^{3/2}} \tag{2.41}$$

as $m \to \infty$.

Step 5. Numerator. We need to show

$$\lim_{n\to\infty} P\Big(\frac{1}{\sqrt{m}}\Big(k_1 - \lceil \frac{n}{m} \rceil - \frac{m}{t_0^2}\Big) \le x\Big) = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{x} e^{-\frac{t^2}{2\sigma^2}}\,dt$$

for every $x \in \mathbb{R}$, where $\sigma = \frac{1}{\sqrt{|\psi''(t_0)|}}$. Recall $\gamma = t_0^{-2}$. By (2.16),

$$P\Big(\frac{1}{\sqrt{m}}\Big(k_1 - \lceil \frac{n}{m} \rceil - \frac{m}{t_0^2}\Big) \le x\Big) = \frac{\sum_{l=1}^{b'_m} q^l \cdot |\mathcal{P}_{ml-j}(m-1)|}{\sum_{l=1}^{M_n+1} q^l \cdot |\mathcal{P}_{lm-j}(m-1)|} \tag{2.42}$$

where $b'_m = [\gamma m + \sqrt{m}\,x] + 1$. Recall $\sqrt{c} \le (4K)^{-1}\psi(t_0)$ be as before. It is known from (2.40) that

$$\sum_{l=1}^{cm} q^l \cdot |\mathcal{P}_{lm-j}(m-1)| \le e^{\psi(t_0)m/2} \tag{2.43}$$

as n is large. Let Ω_1 and Ω_2 be as in (2.21). Set $\Omega_2' = \{l \in \mathbb{N};\, \gamma m - \sqrt{m}\log m \le l \le b'_m\}$. Notice $\Omega_2' \subset \Omega_2$ for large m. By (2.27), (2.31) and (2.43),

$$\begin{aligned}
&\sum_{l=1}^{b'_m} q^l \cdot |\mathcal{P}_{ml-j}(m-1)| \\
=\ & O\big(e^{\psi(t_0)m/2}\big) + \sqrt{m}e^{m\psi(t_0)-(L/2)(\log m)^2} + \sum_{l\in\Omega_2'} q^l \cdot |\mathcal{P}_{ml-j}(m-1)| \\
=\ & O\Big(\sqrt{m}\cdot e^{m\psi(t_0)-(L/2)(\log m)^2}\Big) + \\
& \qquad \frac{t_0^2 f(t_0)}{m^2} e^{\lambda t_0^{-2}-(\lambda j/m)} \sum_{l\in\Omega_2'} \exp\Big\{(m-1)\Big(\frac{g(t_l)}{t_l} - \frac{\lambda}{t_l^2}\Big)\Big\}
\end{aligned} \tag{2.44}$$

as $m \to \infty$. Review the derivation between (2.33) and (2.38) and replace b_m by b'_m. by the fact $\Omega'_2 \subset \Omega_2$ for large m again, we have

$$\begin{aligned}
& e^{-(m-1)\psi(t_0)} \sum_{l \in \Omega'_2} \exp\Big[(m-1)\Big(\frac{g(t_l)}{t_l} - \frac{\lambda}{t_l^2}\Big)\Big] \\
= & \int_{a_m}^{b'_m} \rho(x)\, dx + \epsilon_m + O\big(\frac{\log m}{m^{1/4}}\big)
\end{aligned}$$

where, as mentioned before, $a_m = \gamma m - \sqrt{m}\log m$ and $|\epsilon_m| \le e^{-C_1(\log m)^2}$ for large m. Let us evaluate the integral above. In fact, from (2.34) we see that

$$\int_{a_m}^{b'_m} \rho(x)\, dx = \int_{a_m}^{b'_m} \exp\Big[\frac{1}{2}\psi''(t_0)\big(\frac{c_m}{\sqrt{x}} - t_0\big)^2 m\Big]\, dx.$$

Set $w = -\big(\frac{c_m}{\sqrt{x}} - t_0\big)\sqrt{m}$. Then

$$\begin{aligned}
\int_{a_m}^{b'_m} \rho(x)\, dx &= \frac{2c_m^2}{\sqrt{m}} \int_{-\gamma_m + o(1)}^{\frac{x}{2\gamma^{3/2}} + o(1)} \Big(-\frac{w}{\sqrt{m}} + t_0\Big)^{-3} e^{-\frac{1}{2}|\psi''(t_0)|w^2}\, dw \\
&= (1+o(1))\frac{2\sqrt{m}}{t_0^3} \int_{-\infty}^{\frac{x}{2\gamma^{3/2}}} e^{-\frac{1}{2}|\psi''(t_0)|w^2}\, dw \\
&= (1+o(1))\frac{\sqrt{m}}{t_0^3\gamma^{3/2}} \int_{-\infty}^{x} e^{-w^2/(2\sigma^2)}\, dw = (1+o(1))\sqrt{m}\int_{-\infty}^{x} e^{-w^2/(2\sigma^2)}\, dw
\end{aligned}$$

where $\gamma_m = (\log m)\gamma^{-3/2}/2$ and $\sigma^2 = \frac{4\gamma^3}{|\psi''(t_0)|}$. Collect the assertions from (2.44) to the above to obtain

$$\begin{aligned}
& \sum_{l=1}^{b'_m} q^l \cdot |\mathcal{P}_{ml-j}(m-1)| \\
= & (1+o(1))\frac{t_0^2 f(t_0)}{m^2} e^{\lambda t_0^{-2} - (\lambda j/m)} \cdot e^{(m-1)\psi(t_0)} \cdot \sqrt{m}\int_{-\infty}^{x} e^{-w^2/(2\sigma^2)}\, dw \\
\sim & t_0^2 f(t_0) \cdot e^{\lambda t_0^{-2} - \psi(t_0) - (\lambda j/m)} \cdot \frac{e^{m\psi(t_0)}}{m^{3/2}} \int_{-\infty}^{x} e^{-w^2/(2\sigma^2)}\, dw
\end{aligned}$$

as $m \to \infty$. Join this with (2.41) and (2.42) to conclude that

$$P\Big(\frac{1}{\sqrt{m}}\Big(k_1 - \lceil\frac{n}{m}\rceil - \frac{m}{t_0^2}\Big) \le x\Big) \to \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{x} e^{-w^2/(2\sigma^2)}\, dw \tag{2.45}$$

as $m \to \infty$. Notice that $\sigma^2 = \frac{4}{|\psi''(t_0)|t_0^6}$. The proof is completed by using Lemma 2.2 and the fact $\gamma = t_0^{-2}$. $\square$

3 Proofs of the generalized distribution

3.1 Case I: m is fixed

From Erdös and Lehner (1941), we have

$$\mathcal{P}_n(m) \sim \frac{\binom{n-1}{m-1}}{m!} \tag{3.1}$$

uniformly for $m = o(n^{1/3})$.

Proof of Theorem 3. To prove the conclusion, it suffices to show that for any bounded and Lipschitz continuous function ψ on $\overline{\nabla}_{m-1}$,

$$\mathbb{E}\left(\psi(\frac{k_1}{n}, \dots, \frac{k_m}{n})\right) \to \mathbb{E}\left(\psi(x_1, \dots, x_m)\right)$$

as n tends to infinity. By definition,

$$\begin{aligned}\mathbb{E}\left(\psi(\frac{k_1}{n}, \dots, \frac{k_m}{n})\right) &= \frac{\sum_{(k_1,\dots,k_m)\in\mathcal{P}_n(m)} \psi(\frac{k_1}{n}, \dots, \frac{k_m}{n}) f(\frac{k_1}{n}, \dots, \frac{k_m}{n})}{\sum_{(k_1,\dots,k_m)\in\mathcal{P}_n(m)} f(\frac{k_1}{n}, \dots, \frac{k_m}{n})} \\ &= \frac{n^{-(m-1)}\sum_{(k_1,\dots,k_m)\in\mathcal{R}_n(m)} \psi(\frac{k_1}{n}, \dots, \frac{k_m}{n}) f(\frac{k_1}{n}, \dots, \frac{k_m}{n})}{n^{-(m-1)}\sum_{(k_1,\dots,k_m)\in\mathcal{P}_n(m)} f(\frac{k_1}{n}, \dots, \frac{k_m}{n})} + \mathcal{E}_{n,m},\end{aligned} \tag{3.2}$$

where the set

$$\mathcal{R}_n(m) := \{(k_1, \dots, k_m) \vdash n; k_1 > \dots > k_m > 0\}$$

and

$$\mathcal{E}_{n,m} := \frac{\sum_{(k_1,\dots,k_m)\in\mathcal{P}_n(m)\setminus\mathcal{R}_n(m)} \psi(\frac{k_1}{n}, \dots, \frac{k_m}{n}) f(\frac{k_1}{n}, \dots, \frac{k_m}{n})}{\sum_{(k_1,\dots,k_m)\in\mathcal{P}_n(m)} f(\frac{k_1}{n}, \dots, \frac{k_m}{n})}.$$

On the other hand,

$$\begin{aligned}\mathbb{E}(\psi(x_1, \dots, x_m)) &= \int_{\nabla_{m-1}} \psi(y_1, \dots, y_m) f(y_1, \dots, y_m)\, dy_1 \dots dy_{m-1} \\ &= \frac{\int_{\nabla_{m-1}} \psi(y_1, \dots, y_m) f(y_1, \dots, y_m)\, dy_1 \dots dy_{m-1}}{\int_{\nabla_{m-1}} f(y_1, \dots, y_m)\, dy_1 \dots dy_{m-1}}.\end{aligned} \tag{3.3}$$

In order to compare (3.2) and (3.3), we divide the proof into a few steps.

Step 1: Estimate of $|\mathcal{E}_{n,m}|$. We claim that the term $\mathcal{E}_{n,m}$ is negligible as $n \to \infty$. We first estimate the size of $\mathcal{R}_n(m)$. For any $(k_1, \cdots, k_m) \in \mathcal{R}_n(m)$, set $j_i = k_i - (m - i + 1)$ for $1 \le i \le m$. It is easy to verify that $j_{i-1} - j_i = k_{i-1} - k_i - 1 \ge 0$ for $2 \le i \le m$. Thus

$$j_1 + \dots + j_m = n - \binom{m+1}{2}$$

and $j_1 \geq \cdots \geq j_m \geq 0$. Therefore, $(j_1, \cdots, j_m) \in \mathcal{P}_{n-\binom{m+1}{2}}(m)$. Indeed, this transform is a bijection between $\mathcal{R}_n(m)$ and $\mathcal{P}_{n-\binom{m+1}{2}}(m)$, which implies

$$|\mathcal{R}_n(m)| = |\mathcal{P}_{n-\binom{m+1}{2}}(m)|.$$

On the other hand, we know from (3.1),

$$|\mathcal{P}_N(m)| \sim \frac{\binom{N-1}{m-1}}{m!}$$

as $N \to \infty$. Thus by Stirling's formula,

$$\begin{aligned}
\frac{|\mathcal{R}_n(m)|}{|\mathcal{P}_n(m)|} \quad &\sim \quad \frac{\binom{n-\binom{m+1}{2}-1}{m-1}}{\binom{n-1}{m-1}} = \frac{(n-\binom{m+1}{2}-1)!(n-m)!}{(n-1)!(n-\binom{m+1}{2}-m)!} \\
&\sim \quad \frac{(n-\binom{m+1}{2})!(n-m)!}{n!(n-\binom{m+1}{2}-m)!} \\
&\sim \quad \frac{(1-\frac{m}{n})^{1/2}}{(1-\frac{m}{n-\binom{m+1}{2}})^{1/2}} \frac{(1-\frac{m}{n})^n(1-\frac{\binom{m+1}{2}}{n-m})^m}{(1-\frac{m}{n-\binom{m+1}{2}})^{n-\binom{m+1}{2}}}
\end{aligned}$$

as $n \to \infty$. By assumption $m = o(\sqrt{n})$, we have $\frac{n-\binom{m+1}{2}}{m} \to \infty$ with n. Using the fact that $\lim_{N\to\infty}(1+\frac{x}{N})^N = \exp(x)$, we obtain

$$\frac{|\mathcal{R}_n(m)|}{|\mathcal{P}_n(m)|} \sim \exp\left(-\frac{m\binom{m+1}{2}}{n-m}\right).$$

Thus as long as $m = o(n^{1/3})$,

$$\begin{aligned}
&|\mathcal{R}_n(m)| \sim |\mathcal{P}_n(m)| \\
&|\mathcal{P}_n(m) \setminus \mathcal{R}_n(m)| = o(|\mathcal{P}_n(m)|)
\end{aligned}$$

as $n \to \infty$.

Further, since $\int_{\nabla_{m-1}} f(y_1, \ldots, y_m)\, dy_1 \ldots dy_{m-1} = 1$, there exists a region $\mathcal{S}$ on $\overline{\nabla}_{m-1}$ whose measure $|\mathcal{S}| \geq \mu|\nabla_{m-1}|$ for some constant $\mu > 0$ such that $f(y_1, \ldots, y_m) > c$ on $\mathcal{S}$ for some $c > 0$. By the Lipschitz property of f, for n sufficiently large, $f(k_1/n, \ldots, k_m/n) > c_0 > 0$ for $(k_1, \ldots, k_m)$ in a subset of $\mathcal{P}_n(m)$ with cardinality at least a small fraction of $|\mathcal{P}_n(m)|$. Also since the functions ψ and f are bounded on ∇_{m-1}, we conclude

$$|\mathcal{E}_{n,m}| = O\left(\frac{|\mathcal{P}_n(m) \setminus \mathcal{R}_n(m)|}{|\mathcal{P}_n(m)|}\right) = o(1) \tag{3.4}$$

as $n \to \infty$, as long as $m = o(n^{1/3})$.

Step 2: Compare the numerators of (3.2) *and* (3.3). For convenience, denote

$$G(y_1,\dots,y_{m-1}) = \psi(y_1,\dots,y_{m-1},1-\sum_{i=1}^{m-1} y_i) f(y_1,\dots,y_{m-1},1-\sum_{i=1}^{m-1} y_i).$$

Since ψ, f are bounded and Lipschitz functions on $\overline{\nabla}_{m-1}$, it is easy to check that G is also bounded and Lipschitz on $\overline{\nabla}_{m-1}$. We can rewirte the numberator in (3.2) as follows.

$$\begin{aligned}
\mathcal{I}_1 &:= \frac{1}{n^{m-1}} \sum_{\substack{k_1>\ldots>k_m>0\\ k_1+\ldots+k_m=n}} G(\frac{k_1}{n},\dots,\frac{k_{m-1}}{n}) \\
&= \frac{1}{n^{m-1}} \sum_{(k_1,\ldots,k_{m-1})\in\{1,\ldots,n\}^{m-1}} G(\frac{k_1}{n},\dots,\frac{k_{m-1}}{n}) I_{\mathcal{A}_n} \\
&= \sum_{(k_1,\ldots,k_{m-1})\in\{1,\ldots,n\}^{m-1}} \int_{\frac{k_1-1}{n}}^{\frac{k_1}{n}} \cdots \int_{\frac{k_{m-1}-1}{n}}^{\frac{k_{m-1}}{n}} G(\frac{k_1}{n},\dots,\frac{k_{m-1}}{n}) I_{\mathcal{A}_n}\ dy_1\dots dy_{m-1},
\end{aligned}$$

where $I_{\mathcal{A}_n}$ is the indicator function of set $\mathcal{A}_n$ defined as below

$$\mathcal{A}_n = \frac{1}{n}\Big\{(k_1,\cdots,k_{m-1}) \in \{1,\dots,n\}^{m-1};\ \frac{k_1}{n} > \cdots > \frac{k_{m-1}}{n} > 1-\sum_{i=1}^{m-1}\frac{k_i}{n} > 0\Big\}. \tag{3.5}$$

Similarly,

$$\begin{aligned}
\mathcal{I}_2 &:= \int_{\nabla_{m-1}} G(y_1,\dots,y_{m-1})\, dy_1\dots dy_{m-1} \\
&= \int_{[0,1]^{m-1}} G(y_1,\dots,y_{m-1}) I_{\mathcal{A}}\, dy_1\dots dy_{m-1} \\
&= \sum_{(k_1,\ldots,k_{m-1})\in\{1,\ldots,n\}^{m-1}} \int_{\frac{k_1-1}{n}}^{\frac{k_1}{n}} \cdots \int_{\frac{k_{m-1}-1}{n}}^{\frac{k_{m-1}}{n}} G(y_1,\dots,y_{m-1}) I_{\mathcal{A}}\ dy_1\dots dy_{m-1},
\end{aligned}$$

where the $I_{\mathcal{A}}$ is the indicator function of set $\mathcal{A}$ denoted by

$$\mathcal{A} = \Big\{(x_1,\cdots,x_{m-1}) \in [0,1]^{m-1};\ x_1 > \cdots > x_{m-1} > 1-\sum_{i=1}^{m-1} x_i \ge 0\Big\}. \tag{3.6}$$

Now we estimate the difference between the numerators in (3.2) and (3.3).

$$\begin{aligned}
&\mathcal{I}_1 - \mathcal{I}_2 \\
= &\sum_{(k_1,\ldots,k_{m-1})\in\{1,\ldots,n\}^{m-1}} \int_{\frac{k_1-1}{n}}^{\frac{k_1}{n}} \cdots \int_{\frac{k_{m-1}-1}{n}}^{\frac{k_{m-1}}{n}} \\
&\qquad \Big(G(\frac{k_1}{n},\dots,\frac{k_{m-1}}{n}) I_{\mathcal{A}_n} - G(y_1,\dots,y_{m-1}) I_{\mathcal{A}}\Big)\ dy_1\dots dy_{m-1}
\end{aligned}$$

which is identical to

$$
\begin{aligned}
&\sum_{(k_1,\ldots,k_{m-1})\in\{1,\ldots,n\}^{m-1}} \int_{\frac{k_1-1}{n}}^{\frac{k_1}{n}} \cdots \int_{\frac{k_{m-1}-1}{n}}^{\frac{k_{m-1}}{n}} \\
&\qquad \left(G(\frac{k_1}{n},\ldots,\frac{k_{m-1}}{n}) - G(y_1,\ldots,y_{m-1}) \right) I_{\mathcal{A}_n}\ dy_1\ldots dy_{m-1} \\
&\quad + \sum_{(k_1,\ldots,k_{m-1})\in\{1,\ldots,n\}^{m-1}} \int_{\frac{k_1-1}{n}}^{\frac{k_1}{n}} \cdots \int_{\frac{k_{m-1}-1}{n}}^{\frac{k_{m-1}}{n}} \\
&\qquad G(y_1,\ldots,y_{m-1})\,(I_{\mathcal{A}_n} - I_{\mathcal{A}})\ dy_1\ldots dy_{m-1} \\
&:= \mathcal{S}_1 + \mathcal{S}_2.
\end{aligned}
$$

Step 3: Estimate $\mathcal{S}_1$. Since G is Lipschitz, for $y_i \in [\frac{k_i-1}{n}, \frac{k_i}{n}]$ $(1 \le i \le m-1)$,

$$
\begin{aligned}
|G(\frac{k_1}{n},\ldots,\frac{k_{m-1}}{n}) - G(y_1,\ldots,y_{m-1})| &\le C\cdot\sqrt{\sum_{i=1}^{m-1}(y_i - \frac{k_i}{n})^2} \\
&\le C\cdot\frac{\sqrt{m}}{n},
\end{aligned}
$$

for some constant C depending only on the Lipschitz constant of G. Thus

$$
\begin{aligned}
|\mathcal{S}_1| &\le \sum_{(k_1,\ldots,k_{m-1})\in\{1,\ldots,n\}^{m-1}} \int_{\frac{k_1-1}{n}}^{\frac{k_1}{n}} \cdots \int_{\frac{k_{m-1}-1}{n}}^{\frac{k_{m-1}}{n}} \\
&\qquad |G(\frac{k_1}{n},\ldots,\frac{k_{m-1}}{n}) - G(y_1,\ldots,y_{m-1})|\ dy_1\ldots dy_{m-1} \\
&\le C\cdot\frac{\sqrt{m}}{n}\left(\frac{1}{n}\right)^{m-1} n^{m-1} = \frac{C\sqrt{m}}{n}.
\end{aligned} \tag{3.7}
$$

Step 4: Estimate $\mathcal{S}_2$. Since G is bounded on $\overline{\nabla}_{m-1}$, $\|G\|_\infty := \sup_{\mathbf{x}\in\overline{\nabla}_{\mathbf{m-1}}} |G(\mathbf{x})| < \infty$ and thus

$$
|\mathcal{S}_2| \le \|G\|_\infty \sum_{(k_1,\ldots,k_{m-1})\in\{1,\ldots,n\}^{m-1}} \int_{\frac{k_1-1}{n}}^{\frac{k_1}{n}} \cdots \int_{\frac{k_{m-1}-1}{n}}^{\frac{k_{m-1}}{n}} |I_{\mathcal{A}_n} - I_{\mathcal{A}}|\ dy_1\ldots dy_{m-1}. \tag{3.8}
$$

Now we control $|I_{\mathcal{A}_n} - I_{\mathcal{A}}|$ provided $\frac{k_i-1}{n} < y_i < \frac{k_i}{n}$ for $1 \le i \le m-1$. By definition,

$$
I_{\mathcal{A}_n} = \begin{cases} 1, \text{if } \frac{k_1}{n} > \cdots > \frac{k_{m-1}}{n} > 1 - \sum_{i=1}^{m-1}\frac{k_i}{n} > 0 \\ 0, \text{otherwise} \end{cases} \tag{3.9}
$$

and

$$
I_{\mathcal{A}} = \begin{cases} 1, \text{if } y_1 > \cdots > y_{m-1} > 1 - \sum_{i=1}^{m-1} y_i \ge 0 \\ 0, \text{otherwise.} \end{cases} \tag{3.10}
$$

Let $\mathcal{B}_n$ be a subset of $\mathcal{A}_n$ such that

$$\mathcal{B}_n = \mathcal{A}_n \cap \Big\{(k_1, \cdots, k_{m-1}) \in \{1, \dots, n\}^{m-1};\ \frac{k_{m-1}}{n} + \sum_{i=1}^{m-1} \frac{k_i}{n} > \frac{m}{n} + 1\Big\}.$$

Given $(k_1, \cdots, k_{m-1}) \in \mathcal{B}_n$, for any

$$\frac{k_1 - 1}{n} < y_1 < \frac{k_1}{n}, \cdots, \frac{k_{m-1} - 1}{n} < y_{m-1} < \frac{k_{m-1}}{n}, \tag{3.11}$$

it is easy to verify from (3.10) and (3.9) that $I_{\mathcal{A}} = 1$. Hence,

$$\begin{aligned} I_{\mathcal{A}_n} &= I_{\mathcal{B}_n} + I_{\mathcal{A}_n \setminus \mathcal{B}_n} \\ &\le I_{\mathcal{A}} + I_{\mathcal{A}_n \cap \{k_{m-1} + \sum_{i=1}^{m-1} k_i \le n+m\}} \\ &= I_{\mathcal{A}} + \sum_{j=n+1}^{n+m} I_{E_j} \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} E_j &= \Big\{(k_1, \cdots, k_{m-1}) \in \{1, \dots, n\}^{m-1};\ k_1 > \dots > k_{m-1} \ge 1, \\ &\qquad\qquad k_{m-1} + \sum_{i=1}^{m-1} k_i = j, \sum_{i=1}^{m-1} k_i < n\Big\} \end{aligned}$$

for $n + 1 \le j \le m + n$. Let us estimate the size of $|E_j|$. From the last two restrictions, we obtain $k_{m-1} > j - n$. Since $\sum_{i=1}^{m-1} k_i < n$ and $k_i > k_{m-1}$ for $1 \le i \le m - 2$, we have $j - n + 1 \le k_{m-1} \le \frac{n}{m-1}$.

For each fixed k_{m-1}, since $k_1 > \dots > k_{m-2}$ is the ordered positive integer solution to the linear equation $\sum_{i=1}^{m-2} k_i = j - 2k_{m-1}$, thus

$$|E_j| \le \sum_{j-n+1 \le l \le \frac{n}{m-1}} \frac{\binom{j-2l-1}{m-3}}{(m-2)!} \le \left(\frac{n}{m-1} + n - j\right) \frac{\binom{2n-j-3}{m-3}}{(m-2)!}.$$

As a result, we obtain the crude upper bound

$$\sum_{j=n+1}^{n+m} |E_j| \le \sum_{j=n+1}^{n+m} \left(\frac{n}{m-1} + n - j\right) \frac{\binom{2n-j-3}{m-3}}{(m-2)!} \le \frac{m \cdot n^{m-2}}{(m-1)!(m-3)!}. \tag{3.13}$$

On the other hand, consider a subset of $\mathcal{A}_n^c := \{\frac{1}{n}, \frac{2}{n}, \cdots, 1\}^{m-1} \setminus \mathcal{A}_n$ defined by

$$\begin{aligned} \mathcal{C}_n &= \frac{1}{n}\Big\{(k_1, \cdots, k_{m-1}) \in \{1, \dots, n\}^{m-1}; \text{ either } k_i \le k_{i+1} - 1 \text{ for some } 1 \le i \le m-2, \\ &\quad \text{or } k_1 + \cdots + k_{m-2} + 2k_{m-1} \le n, \text{ or } k_1 + \cdots + k_{m-1} \ge m + n - 1\Big\}. \end{aligned}$$

Set $\mathcal{A}^c = [0,1]^{m-1} \backslash \mathcal{A}$. Given $(\frac{k_1}{n}, \cdots, \frac{k_{m-1}}{n}) \in \mathcal{C}_n$, for any k_i's and y_i's satisfying (3.11), it is not difficult to check that $I_{\mathcal{A}^c} = 1$. Consequently,

$$\begin{aligned} I_{\mathcal{A}_n^c} &= I_{\mathcal{C}_n} + I\Big\{(\frac{k_1}{n}, \cdots, \frac{k_{m-1}}{n}) \in \mathcal{A}_n^c;\ k_i > k_{i+1} - 1 \text{ for all } \ 1 \le i \le m-2, \\ &\qquad k_1 + \cdots + k_{m-2} + 2k_{m-1} > n, \text{ and } k_1 + \cdots + k_{m-1} < m + n - 1\Big\} \\ &\le I_{\mathcal{A}^c} + I(\mathcal{D}_{n,m,1}) + I(\mathcal{D}_{n,m,2}), \end{aligned}$$

or equivalently,

$$I_{\mathcal{A}_n} \ge I_{\mathcal{A}} - I(\mathcal{D}_{n,m,1}) - I(\mathcal{D}_{n,m,2}), \tag{3.14}$$

where

$$\mathcal{D}_{n,m,1} = \bigcup_{l=n}^{n+m-2} \frac{1}{n}\big\{(k_1, \cdots, k_{m-1}) \in \{1, \ldots, n\}^{m-1};\ \sum_{i=1}^{m-1} k_i = l, k_1 \ge \ldots \ge k_{m-1}\big\};$$

$$\begin{aligned} \mathcal{D}_{n,m,2} = \bigcup_{l=1}^{m-2} \frac{1}{n}\big\{&(k_1, \cdots, k_{m-1}) \in \{1, \ldots, n\}^{m-1};\ k_l = k_{l+1}, k_1 \ge \ldots \ge k_{m-1}, \\ &\sum_{i=1}^{m-1} k_i + k_{m-1} \ge n+1, \sum_{i=1}^{m} k_i \le n + m - 2\big\}; \end{aligned}$$

By the definition of partitions and (3.1), we have the following bound on $|\mathcal{D}_{n,m,1}|$.

$$\begin{aligned} |\mathcal{D}_{n,m,1}| &\le \sum_{l=n}^{n+m-2} |\mathcal{P}_l(m-1)| \sim \sum_{l=n}^{n+m-2} \frac{\binom{l-1}{m-2}}{(m-1)!} \\ &\le (m-1)\frac{\binom{n+m-2}{m-2}}{(m-1)!} \le \frac{(n+m-2)^{m-2}}{[(m-2)!]^2} \end{aligned} \tag{3.15}$$

as $n \to \infty$.

The estimation of $|\mathcal{D}_{n,m,2}|$ is the same argument as in (3.13). For the cases $m = 3$ or $m = 4$, it is easy to verify that $|\mathcal{D}_{n,m,2}| = O(n^{m-2})$. Now we assume $m \ge 5$. First, from the decreasing order of k_i and $\sum_{i=1} k_i \le n + m - 2$, we determine the range of k_{m-1},

$$1 \le k_{m-1} \le \frac{n+m-2}{m-1}.$$

On the other hand, $n + 1 - 2k_{m-1} \le \sum_{i=1}^{m-2} k_i \le n + m - 2 - k_{m-1}$. If $l \ne m-2$, from the restriction $k_l = k_{l+1}$, we see $k_1 + \ldots + k_{l-1} + k_{l+2} + \ldots + k_{m-2} = s - 2k_l$ is the ordered positive integer solutions to the equation $j_1 + \ldots + j_{m-4} = s - 2k_l$, where $n + 1 - 2k_{m-1} \le s \le n + m - 2 - k_{m-1}$. If $l = m-2$, then $k_1 + \cdots + k_{m-3} = s - 2k_{m-1}$ and $n + 1 - 3k_{m-1} \le s - 2k_{m-1} \le n + m - 2 - 2k_{m-1}$. Therefore, we have the following crude upper bound

$$\begin{aligned}|\mathcal{D}_{n,m,2}| &\leq \sum_{l=1}^{m-3}\sum_{k_{m-1}=1}^{\frac{n+m-2}{m-1}}\sum_{s=n+1-2k_{m-1}}^{n+m-2-k_{m-1}}\sum_{k_{m-1}\leq k_l\leq s/2}\frac{\binom{s-2k_l-1}{m-5}}{(m-4)!}\\
&\qquad+\sum_{k_{m-1}=1}^{\frac{n+m-2}{m-1}}\sum_{s=n+1-3k_{m-1}}^{n+m-2-2k_{m-1}}\frac{\binom{s-k_{m-1}-1}{m-4}}{(m-3)!}\\
&= O\left(\frac{n^3(m-3)}{m^2(m-4)!}\binom{n+m-6}{m-5}+\frac{n^2}{m^2(m-3)!}\binom{n+m-6}{m-4}\right)\\
&= O\left(\frac{n^2(n+m)^{m-4}}{m(m-4)!(m-5)!}\right). \qquad (3.16)\end{aligned}$$

Joining (3.12) and (3.14), and assuming (3.11) holds, we arrive at

$$|I_{\mathcal{A}_n}-I_{\mathcal{A}}|\leq I(\mathcal{D}_{n,m,1})+I(\mathcal{D}_{n,m,2})+\sum_{i=n+1}^{n+m}I_{E_i}$$

Observe that $\mathcal{D}_{n,m,i}$'s and E_i's do not depend on x_i's, we obtain from (3.8) that

$$\begin{aligned}|\mathcal{S}_2| &\leq \|G\|_\infty\sum_{k_1=1}^{n}\cdots\sum_{k_{m-1}=1}^{n}\Big[\sum_{i=1}^{2}I(\mathcal{D}_{n,m,i})+\sum_{i=n}^{n+m}I_{E_i}\Big]\int_{\frac{k_1-1}{n}}^{\frac{k_1}{n}}\cdots\int_{\frac{k_{m-1}-1}{n}}^{\frac{k_{m-1}}{n}}1\,dx_1\ldots dx_{m-1}\\
&= \|G\|_\infty\Big(\sum_{i=1}^{2}|\mathcal{D}_{n,m,i}|+\sum_{i=n}^{n+m}|E_i|\Big)\cdot\frac{1}{n^{m-1}}.\end{aligned}$$

For $2\leq m\leq 4$,

$$|\mathcal{S}_2|=O(n^{-1}).$$

For $m\geq 5$, by (3.13),(3.15) and (3.16),

$$\begin{aligned}|\mathcal{S}_2| &= O\left(\frac{m\cdot n^{m-2}}{(m-1)!(m-3)!}+\frac{(n+m)^{m-2}}{[(m-2)!]^2}+\frac{n^2(n+m)^{m-4}}{m(m-4)!(m-5)!}\right)\cdot\frac{1}{n^{m-1}}\\
&= O\left(\frac{(1+\frac{m}{n})^m}{n}\right)\end{aligned}$$

as $n\to\infty$.

Step 5: Difference between the expectations (3.2) *and* (3.3). From *Step 3* and *Step 4*, we obtain the difference between the numberators in (3.2) and (3.3)

$$|\mathcal{I}_1-\mathcal{I}_2|\leq|\mathcal{S}_1|+|\mathcal{S}_2|\leq C_1\cdot\left(\frac{\sqrt{m}}{n}+\frac{(1+\frac{m}{n})^m}{n}\right) \qquad (3.17)$$

as $n\to\infty$ for some constant C_1 depending only on the Lipschitz constants of ψ and f and the upper bounds of ψ and f on the compact set $\overline{\nabla}_{m-1}$. Choosing ψ to be identity on $\overline{\nabla}_{m-1}$, we can bound the denominators in (3.2) and (3.3).

Finally, we estimate the expectations (3.2) and (3.3). Since m is fixed, by (3.4) and (3.17),

$$\begin{aligned} \left|\mathbb{E}\left(\psi(\frac{k_1}{n},\dots,\frac{k_m}{n})\right) - \mathbb{E}\left(\psi(x_1,\dots,x_m)\right)\right| &= O(\frac{\sqrt{m}}{n} + \frac{(1+\frac{m}{n})^m}{n}) + |\mathcal{E}_{n,m}| \quad (3.18) \\ &\to 0 \end{aligned}$$

as $n \to \infty$. □

Proof of Corollary 2. By Theorem 3,

$$(\frac{k_1}{n},\dots,\frac{k_m}{n}) \to (x_1,\dots,x_m) \sim \mu$$

as $n \to \infty$, where μ has pdf

$$g(y_1,\dots,y_m) = \frac{y_1^{\alpha-1}\cdots y_m^{\alpha-1}}{\int_{\nabla_{m-1}} y_1^{\alpha-1}\cdots y_m^{\alpha-1}\,dy_1\dots dy_{m-1}}. \tag{3.19}$$

It suffices to show the order statistics $(X_{(1)},\dots,X_{(m)})$ of $(X_1,\dots,X_m) \sim \mathrm{Dir}(\alpha)$ has the same pdf on ∇_{m-1}. For any continuous function ψ defined on ∇_{m-1}, by symmetry,

$$\begin{aligned} &\mathbb{E}\psi(X_{(1)},\dots,X_{(m)}) \\ = &\int_{W_{m-1}} \psi(y_{(1)},\dots,y_{(m)})\mathbf{1}_{\{y_{(1)}\geq\dots\geq y_{(m)}\}}\frac{\Gamma(m\alpha)}{\Gamma(\alpha)^m}y_1^{\alpha-1}\cdots y_m^{\alpha-1}\,dy_1\dots dy_{m-1} \\ = &\int_{W_{m-1}} \sum_{\sigma\in\mathcal{S}_m}\psi(y_{\sigma(1)},\dots,y_{\sigma(m)})\mathbf{1}_{\{y_{\sigma(1)}\geq\dots\geq y_{\sigma(m)}\}}\frac{\Gamma(m\alpha)}{\Gamma(\alpha)^m}y_{\sigma(1)}^{\alpha-1}\cdots y_{\sigma(m)}^{\alpha-1}\,dy_1\dots dy_{m-1} \\ = &\int_{\nabla_{m-1}} \psi(y_1,\dots,y_m)\frac{m!\Gamma(m\alpha)}{\Gamma(\alpha)^m}y_1^{\alpha-1}\cdots y_m^{\alpha-1}\,dy_1\dots dy_{m-1}. \end{aligned}$$

Therefore, the pdf of $(X_{(1)},\dots,X_{(m)})$ is

$$\frac{m!\Gamma(m\alpha)}{\Gamma(\alpha)^m}y_1^{\alpha-1}\cdots y_m^{\alpha-1} \tag{3.20}$$

on the set ∇_{m-1}.

Similarly, since by the definition of pdf

$$\int_{W_{m-1}} \frac{\Gamma(m\alpha)}{\Gamma(\alpha)^m}x_1^{\alpha-1}\cdots x_m^{\alpha-1} = 1,$$

by symmetry, we now have

$$\int_{\nabla_{m-1}} y_1^{\alpha-1}\cdots y_m^{\alpha-1}\,dy_1\dots dy_{m-1} = \frac{\Gamma(\alpha)^m}{m!\Gamma(m\alpha)}.$$

Comparing the above with (3.20) and (3.19), we complete the proof. □

Proof of Corollary 3. By Theorem 3 or Corollary 2,

$$(\frac{k_1}{n}, \ldots, \frac{k_m}{n}) \to (Y_1, \ldots, Y_m) \sim \mu$$

as $n \to \infty$, where μ has pdf

$$\frac{m! \cdot \Gamma(\frac{m}{\alpha})}{\Gamma(\frac{1}{\alpha})^m}(y_1 \ldots y_m)^{\alpha-1}$$

on ∇_{m-1} and zero elsewhere. Since $f(x) = x^\alpha$ is continuous,

$$\left((\frac{k_1}{n})^\alpha, \ldots, (\frac{k_m}{n})^\alpha\right) \to (Y_1^\alpha, \ldots, Y_m^\alpha)$$

as $n \to \infty$.

Now it suffices to show $(Y_1^\alpha, \ldots, Y_m^\alpha)$ has the uniform distribution on the set

$$\mathcal{U}_{m-1} = \{(x_1, \ldots, x_m) \in [0,1]^m; \sum_{i=1}^m x_i^{\frac{1}{\alpha}} = 1, x_1 \geq \ldots \geq x_m\}.$$

This can be seen by change of variables. For any continuous function ψ defined on ∇_{m-1},

$$\begin{aligned}
& \mathbb{E}\psi(Y_1^\alpha, \ldots, Y_m^\alpha) \\
= & \int_{\nabla_{m-1}} \psi(y_1^\alpha, \ldots, y_m^\alpha)\frac{m! \cdot \Gamma(\frac{m}{\alpha})}{\Gamma(\frac{1}{\alpha})^m} y_1^{\alpha-1} \cdots y_m^{\alpha-1}\, dy_1 \ldots dy_{m-1} \\
= & \int_{\mathcal{U}_{m-1}} \psi(x_1, \ldots, x_m)\frac{m! \cdot \Gamma(\frac{m}{\alpha})}{\alpha^{m-1}\Gamma(\frac{1}{\alpha})^m}\, dx_1 \ldots dx_{m-1}.
\end{aligned}$$

In the last equality, we set $x_i = y_i^\alpha$ for $1 \leq i \leq m$. Therefore, we can see the pdf of $(Y_1^\alpha, \ldots, Y_m^\alpha)$ is a constant on $\mathcal{U}_{m-1}$, which is the uniform distribution on $\mathcal{U}_{m-1}$. The proof is complete. □

3.2 Case II: m tends to infinity and $m = o(n^{1/3})$

Now we consider the case that m depends on n. The formula (3.1) holds as long as $m = o(n^{1/3})$.

Let μ and ν be two Borel probability measures on a Polish space S with the Borel σ-algebra $\mathcal{B}(S)$. Define

$$\rho(\mu, \nu) \quad = \quad \sup_{\|\varphi\|_L \leq 1} \left| \int_S \varphi(x)\, \mu(dx) - \int_S \varphi(x)\, \nu(dx) \right|, \qquad (3.21)$$

where φ is a bounded Lipschitz function defined on S with $\|\varphi\| = \sup_{x \in S} |\varphi(x)|$, and $\|\varphi\|_L = \|\varphi\| + \sup_{x \neq y} |\varphi(x) - \varphi(y)|/|x - y|$. It is known that μ_n converges to μ weakly if and only if $\lim_{n\to\infty} \int \varphi(x)\, \mu_n(dx) = \int \varphi(x)\, \mu(dx)$ for every bounded and continuous function $\varphi(x)$ defined on $\mathbb{R}^m$, and if and only if $\lim_{n\to\infty} \rho(\mu_n, \mu) = 0$; see, e.g., Chapter 11 from Dudley (2002).

Let $\{X_i, X_{n,i};\ n \geq 1,\ i \geq 1\}$ be random variables taking values in $[0,1]$. Set $X_n = (X_{n1}, X_{n2}, \cdots) \in [0,1]^\infty$. If $X_{ni} = 0$ for $i > m$, we simply write $X_n = (X_{n1}, \cdots, X_{nm})$. We say that X_n *converges weakly* to $X := (X_1, X_2, \cdots)$ as $n \to \infty$ if, for any $r \geq 1$, $(X_{n1}, \cdots, X_{nr})$ converges weakly to $X = (X_1, \cdots, X_r)$ as $n \to \infty$. This convergence actually is the same as the weak convergence of random variables in $([0,1]^\infty, d)$ where

$$d(x,y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i} \tag{3.22}$$

for $x = (x_1, x_2, \cdots) \in [0,1]^\infty$ and $y = (y_1, y_2, \cdots) \in [0,1]^\infty$. The topology generated by this metric is the same as the product topology.

THEOREM 5. *Let $m = m_n \to \infty$ as $n \to \infty$. Assume $\kappa = (k_1, \ldots, k_m) \in \mathcal{P}_n(m)$ is chosen with probability as in* (1.3)*. Let $(X_{m,1}, \cdots, X_{m,m})$ and $X = (X_1, X_2, \cdots)$ be random variables taking values in ∇_{m-1} and ∇, respectively. If*

$$\sup_{\|\varphi\|_L \leq 1} \Big| E\varphi\Big(\frac{k_1}{n}, \cdots, \frac{k_m}{n}\Big) - E\varphi(X_{m,1}, \cdots, X_{m,m}) \Big| \to 0 \tag{3.23}$$

as $n \to \infty$, and $(X_{m,1}, \cdots, X_{m,m})$ converges weakly to X as $n \to \infty$, then $\big(\frac{k_1}{n}, \cdots, \frac{k_m}{n}\big)$ converges weakly to X as $n \to \infty$.

Proof. Given integer $r \geq 1$, to prove the theorem, it is enough to show $\big(\frac{k_1}{n}, \cdots, \frac{k_r}{n}\big)$ converges weakly to $(X_1, \cdots, X_r)$ as $n \to \infty$. Since $m = m_n \to \infty$ as $n \to \infty$, without loss of generality, we assume $r < m$ in the rest of discussion. For any random vector Z, let $\mathcal{L}(Z)$ denote its probability distribution. Review (3.21). By the triangle inequality,

$$\begin{aligned} & \rho\Big(\mathcal{L}\big(\frac{k_1}{n}, \cdots, \frac{k_r}{n}\big),\, \mathcal{L}\big(X_1, \cdots, X_r\big)\Big) \\ \leq\ & \rho\Big(\mathcal{L}\big(\frac{k_1}{n}, \cdots, \frac{k_r}{n}\big),\, \mathcal{L}\big(X_{m,1}, \cdots, X_{m,r}\big)\Big) + \rho\Big(\mathcal{L}\big(X_{m,1}, \cdots, X_{m,r}\big),\, \mathcal{L}\big(X_1, \cdots, X_r\big)\Big) \end{aligned} \tag{3.24}$$

For any function $\varphi(x_1, \cdots, x_r)$ defined on $[0,1]^r$ with $\|\varphi\|_L \leq 1$, set $\tilde{\varphi}(x_1, \cdots, x_m) = \varphi(x_1, \cdots, x_r)$ for all $(x_1, \cdots, x_m) \in \mathbb{R}^m$. Then $\|\tilde{\varphi}\|_L \leq 1$. Condition (3.23) implies that the middle one among the three distances in (3.24) goes to zero. Further, the assumption that $(X_{m,1}, \cdots, X_{m,m})$ converges weakly to X implies the third distance in (3.24) also goes to zero. Hence the first distance goes to zero. The proof is completed. $\square$

With Theorem 5 and the estimation in Theorem 3, we obtain the proof of Theorem 4 immediately.

Proof of Theorem 4. Assume $\kappa = (k_1, \dots, k_m) \in \mathcal{P}_n(m)$ is chosen with probability as in (1.3). In the proof of Theorem 3, we have shown in (3.18) that for any $\|\varphi\|_L \le 1$,

$$\begin{aligned} & \sup_{\|\varphi\|_L \le 1} |\mathbb{E}\left(\varphi(\frac{k_1}{n}, \dots, \frac{k_m}{n})\right) - \mathbb{E}\left(\varphi(x_1, \dots, x_m)\right)| \\ = \ & O(\frac{\sqrt{m}}{n} + \frac{(1+\frac{m}{n})^m}{n}) + |\mathcal{E}_{n,m}| \to 0. \end{aligned}$$

as $n \to \infty$. Recall in (3.4), we have $|\mathcal{E}_{n,m}| \to 0$ as long as $m = o(n^{1/3})$. Therefore, by Theorem 4, we conclude that $(\frac{k_1}{n}, \cdots, \frac{k_m}{n})$ converges weakly to X as $n \to \infty$. $\square$

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